Section 7.4 – Lagrange Multipliers

2. Optimize $f(x,y) = xy$ if $x^2+y^2=1$.

$F(x,y,\lambda) = xy - \lambda(x^2+y^2-1)$

$F_x = y - 2x\lambda = 0 \longrightarrow {}^{①}\ y = 2x\lambda$

$F_y = x - 2y\lambda = 0 \qquad x - 2(2x\lambda)\lambda = 0$

$F_\lambda = -x^2-y^2+1 = 0 \qquad x - 4x\lambda^2 = 0 = x(1-4\lambda^2)=0$

$x = 0$ or $\lambda = \pm \frac{1}{2}$

then $y=0$, but this can't work b/c it won't satisfy $F_\lambda = 0$.

So we know $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$

If $\lambda = \frac{1}{2}$, then $y = x$ ①. If $\lambda = -\frac{1}{2}$, then $y = -x$ ①

Using F_λ, $y=x$, then $2x^2=1$, $x = \pm\sqrt{\frac{1}{2}}$ $\qquad$ $y=-x$, then $2x^2=1$, $x = \mp\sqrt{\frac{1}{2}}$.

Critical points: $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right), \left(\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right)$.

Fill into original to see that $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$ and $\left(\frac{-1}{\sqrt2}, \frac{-1}{\sqrt2}\right)$ are max

$\left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}\right)$ and $\left(\frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)$ are min.

6. Optimize $f(x,y) = 8x^2 - 24xy + y^2$ subject to $8x^2+y^2=1$.

$F(x,y,\lambda) = 8x^2 - 24xy + y^2 - \lambda(8x^2+y^2-1)$

$F_x = 16x - 24y - 16\lambda x \xrightarrow{\div 8} 2x - 3y - 2\lambda x = 0 = 2x(1-\lambda) - 3y$

$F_y = -24x + 2y - 2\lambda y \xrightarrow{\div 2} -12x + y - \lambda y = 0 = y(1-\lambda) - 12x$

$F_\lambda = -8x^2 - y^2 + 1$

$\left.\begin{aligned} 1-\lambda &= 3y/2x \\ 1-\lambda &= 12x/y \end{aligned}\right\}$ $\quad \frac{3y}{2x} = \frac{12x}{y}$

$3y^2 = 24x^2$

$y^2 = 8x^2$

$-8x^2 - 8x^2 + 1 = 0$

$1 = x^2 \qquad y^2 = 8\left(\frac{1}{16}\right) = \frac{1}{2}$

$x = \pm\frac{1}{4} \qquad y = \pm\frac{1}{\sqrt2}$

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6. $f(\frac{1}{4}, \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{6}{\sqrt{2}} + \frac{1}{2}$ minima
$f(\frac{-1}{4}, \frac{-1}{\sqrt{2}}) = \frac{1}{2} - \frac{6}{\sqrt{2}} + \frac{1}{2}$

$f(\frac{1}{4}, \frac{-1}{\sqrt{2}}) = \frac{1}{2} + \frac{6}{\sqrt{2}} + \frac{1}{2}$ maxima
$f(\frac{-1}{4}, \frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{6}{\sqrt{2}} + \frac{1}{2}$

10. Minimize $f(x,y) = 2x^2 + y^2 + 2xy + 4x + 2y + 7$ subject to $4x^2 + 4xy = 1$.

$F(x,y,\lambda) = 2x^2 + y^2 + 2xy + 4x + 2y + 7 - \lambda(4x^2 + 4xy - 1)$

$F_x = 4x + 2y + 4 - 8x\lambda - 4y\lambda = 0 \xrightarrow{\div 2} 2x + y + 2 - 4x\lambda - 2y\lambda = 0$

$F_y = 2y + 2x + 2 - 4x\lambda = 0 \longrightarrow -(2x + 2y + 2 - 4x\lambda = 0)$

$F_\lambda = -4x^2 - 4xy + 1 = 0$

$-y - 2y\lambda = 0$, $-y(1+2\lambda) = 0$

either $y = 0$ or $\lambda = \frac{-1}{2}$

If $y=0$, use $F_\lambda = 0 = -4x^2 + 1 \to x = \pm\frac{1}{2}$.

$f(\frac{1}{2}, 0) = \frac{1}{2} + 2 + 7$ max at $(\frac{1}{2}, 0)$

$f(\frac{-1}{2}, 0) = \frac{1}{2} - 2 + 7$ $\boxed{\text{min at } (\frac{-1}{2}, 0)}$

If $\lambda = \frac{-1}{2}$ Then $F_x = 0 = 4x + 2y + 4 + 4x + 2y = 8x + 4y + 4 = 0$, so $y = -2x - 1$

$F_\lambda = 0 = -4x - 4x(-2x-1) + 1 = -4x + 8x - 4x + 1 = 1 = 0$

this never happens, so no points from this.

14. Optimize $f(x,y,z) = x + 3y - z$ subject to $z = 2x^2 + y^2$

$F(x,y,z,\lambda) = x + 3y - z - \lambda(z - 2x^2 - y^2)$

$F_x = 1 + 4x\lambda = 0 \longrightarrow 1 - 4x = 0$, so $x = 1/4$

$F_y = 3 + 2y\lambda = 0 \longrightarrow 3 - 2y = 0$, so $y = 3/2$

$F_z = -1 - \lambda = 0 \longrightarrow \lambda = -1$

$F_\lambda = -z + 2x^2 + y^2 = 0 \longrightarrow z = 2(\frac{1}{16}) + \frac{9}{4} = \frac{1}{8} + \frac{18}{8} = \frac{19}{8}$

optimized at $(\frac{1}{4}, \frac{3}{2}, \frac{19}{8})$.

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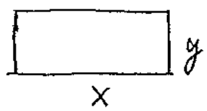
18. 

$2x+2y=320$. Maximize Area $= xy$.

$$F(x,y,\lambda) = xy - \lambda(2x+2y-320).$$

$$\left.\begin{aligned} F_x &= y - 2\lambda = 0 \\ F_y &= x - 2\lambda = 0 \end{aligned}\right\} \; x = y$$

$$F_\lambda = -2x - 2y + 320 = 0 \longrightarrow x+y=160$$

$$2x = 160, \quad x = y = 80\,m.$$

The area is a square, 80 meters per side.

24. Output is $Q(x,y) = 60x^{1/3}y^{2/3}$. Maximize this if $x+y=120$.

$$F(x,y,\lambda) = 60x^{1/3}y^{2/3} - \lambda(x+y-120)$$

$$\left.\begin{aligned} F_x &= 20x^{-2/3}y^{2/3} - \lambda \\ F_y &= 40x^{1/3}y^{-1/3} - \lambda \end{aligned}\right\} \; \lambda = 20x^{-2/3}y^{2/3} = 40x^{1/3}y^{-1/3}$$

$$F_\lambda = -x - y + 120$$

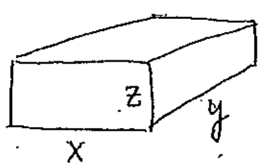
$$x^{2/3}y^{1/3}(20x^{-2/3}y^{2/3}) = x^{2/3}y^{1/3}(40x^{1/3}y^{-1/3})$$

$$20y = 40x$$

$$y = 2x$$

$$-x - 2x + 120 = 0$$

$$3x = 120, \; \boxed{x = 40, \; y = 80}$$

34. 

$V = 16000\ ft^2$. Minimize cost:

$$C = 31xy + 27yz + 27yz + 27xz + 55xz$$

$$F(x,y,z,\lambda) = 31xy + 54yz + 82xz - \lambda(xyz - 16000).$$

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34.

$$F_x = 31y + 82z - yz\lambda = 0 \qquad 31xy + 82xz - xyz\lambda = 0$$
$$F_y = 31x + 54z - xz\lambda = 0 \qquad 31xy + 54yz - xyz\lambda = 0$$
$$F_z = 54y + 82x - xy\lambda = 0 \qquad 54yz + 82xz - xyz\lambda = 0$$
$$F_\lambda = -xyz + 16000. = 0$$

$$xyz\lambda = 31xy + 82xz = 31xy + 54yz = 54yz + 82xz$$

$$82x = 54y \qquad 31y = 82z$$

$$31x = 54z.$$

Fill in to F_λ.

$$-x\left(\tfrac{82}{54}x\right)\left(\tfrac{31}{54}x\right) + 16000. = 0$$
$$16000 = \frac{82 \cdot 31}{54 \cdot 54}x^3$$
$$\frac{16000 \cdot 54 \cdot 54}{82 \cdot 31} = x^3.$$
$$x^3 \approx 18354.05$$
$$x \approx 26.38$$
$$y = \frac{82}{54}x \approx 40.06$$
$$z = \frac{31}{54}x \approx 15.14$$

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36. Constraint: $x+y=8$.

Maximize profit.

P = (# units)(price per unit − cost per unit) − amt spent on dev/promo.

$$P=\left(\frac{320y}{y+2}+\frac{160x}{x+4}\right)(150-50)-8000$$

$$P=\frac{32000y}{y+2}+\frac{16000x}{x+4}-8000$$

$$F(x,y,\lambda)=\frac{32000y}{y+2}+\frac{16000x}{x+4}-8000-\lambda(x+y-8)$$

$$F_x=\frac{(16000)(x+4)-16000x}{(x+4)^2}-\lambda=0$$

$$F_y=\frac{(32000)(y+2)-32000y}{(y+2)^2}-\lambda=0$$

$$F_\lambda=-x-y+8$$

$$\frac{16000(\cancel{x}+4-\cancel{x})}{(x+4)^2}=\lambda=\frac{32000(\cancel{y}+2-\cancel{y})}{(y+2)^2}$$

$$64000/(x+4)^2=64000/(y+2)^2$$

$$(x+4)^2=(y+2)^2$$

$$x+4=\pm(y+2)$$

$$x=\pm(y+2)-4$$

If $x=y+2-4=y-2$, then since $-x-y+8=0$, $(2-y)-y+8=0$

$y=5,\ x=3$

If $x=(-y-2)-4$, then $(-y-6)+y=8$
$=-y-6$

This doesn't work, so only other point is valid.