NAME KEY

Math 12
Test 3
Spring 2010

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Solve $x^2y' + \frac{1}{y^2} = 0$ if $y = 2$ when $x = 1$.

$$x^2y' = \frac{-1}{y^2}$$

$$y' = \frac{-1}{x^2y^2}$$

$$\frac{dy}{dx} = \frac{-1}{x^2y^2}$$

$$y^2dy = -\frac{1}{x^2}dx$$

$$\int y^2dy = -\int x^{-2}dx$$

$$\frac{1}{3}y^3 = x^{-1} + C$$

$y = 2$ when $x = 1$, so

$$\frac{8}{3} = 1 + C$$

$$\frac{5}{3} = C$$

$$\frac{1}{3}y^3 = \frac{1}{x} + \frac{5}{3}$$

$$y^3 = \frac{3}{x} + 5$$

$$y = \sqrt[3]{\frac{3}{x} + 5}$$

2. Find $f'(x)$ for the following functions. DO NOT simplify!

(a) $f(x) = \frac{1+e^x}{1-e^x}$

$$f'(x) = \frac{(e^x)(1-e^x) - (1+e^x)(-e^x)}{(1-e^x)^2}$$

(b) $f(x) = \ln\sqrt{\frac{2x+3}{x^3-4}}$

$$f'(x) = \frac{1}{\sqrt{\frac{2x+3}{x^3-4}}} \cdot \frac{1}{2}\left(\frac{2x+3}{x^3-4}\right)^{-1/2}\left(\frac{(2)(x^3-4) - (2x+3)(3x^2)}{(x^3-4)^2}\right)$$

3. Suppose that Investment A earns interest at an annual rate of 8% compounded quarterly. Investment B earns interest at an annual rate of 7.8% compounded continuously. Which investment has the higher yield? Show your work.

To compare, suppose \$100 is invested for 1 year.

A $\quad B = P\left(1+\frac{r}{k}\right)^{kt}$

$$B = 100\left(1+\frac{.08}{4}\right)^4$$

$$= 100\,(1.02)^4 \approx \$108.24$$

B $\quad B = Pe^{rt}$

$$B = 100\,e^{0.078} \approx \$108.11$$

Investment A has the higher yield.

4. a) Find x if $\log_2 x + 3\log_2 2 = \log_2\left(\frac{2}{x}\right)$

$$\log_2 x + 3 = \log_2 2 - \log_2 x$$

$$\log_2 x + 3 = 1 - \log_2 x$$

$$2\log_2 x = -2$$

$$\log_2 x = -1$$

$$x = 2^{-1} = \frac{1}{2}$$

b) Simplify $e^{\ln x} + \ln e^x + 1$

$$= x + x + 1$$

$$= 2x + 1$$

5. For the function $f(x) = xe^{-x}$, list all intervals of increase and decrease, all maximum and minimum *points*, intervals where the function is concave up and concave down, all inflection *points*, and all asymptotes (or say there are none). Then sketch the graph of the function.

$$f'(x) = (1)(e^{-x}) + (x)(-e^{-x})$$
$$= e^{-x}(1-x) = 0$$

crit #: $x = 1$

max

$+$ | $-$ → f'

1

$$f(1) = (1)(e^{-1}) = e^{-1} = \frac{1}{e}$$

increasing on $(-\infty, 1)$
decreasing on $(1, \infty)$
max at $(1, \frac{1}{e})$
conc up on $(2, \infty)$
conc down on $(-\infty, 2)$
inf pt $(2, \frac{2}{e^2})$
VA: none
HA: $y = 0$

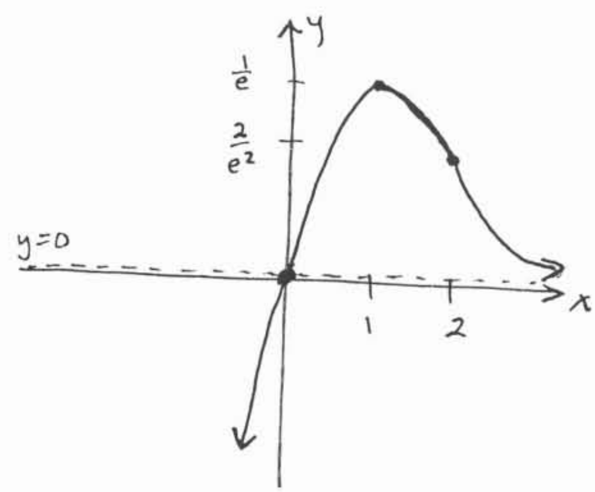
$$f''(x) = (-e^{-x})(1-x) + (e^{-x})(-1)$$
$$= e^{-x}(-1 + x - 1)$$
$$= e^{-x}(x-2) = 0$$

inf #: $x = 2$

$-$ | $+$ → f''

2

$$f(2) = 2e^{-2} = \frac{2}{e^2}$$



f(x) defined for all x, so NO VA
when $x \to \infty$, $e^{-x} \to 0$, so $y \to 0$
$y = 0$ is HA. (on right side)
when $x \to -\infty$, $e^{-x} \to \infty$, $xe^{-x} \to -\infty$
(no HA on left side)

inc down	dec down	dec up

→ combo

1, 2

6. Evaluate the following integrals:

a) $\int \frac{x^3}{e^{x^4}}\,dx \quad = \quad \int \left(\frac{1}{4}\,du\right)\left(\frac{1}{e^u}\right) = \frac{1}{4}\int e^{-u}\,du$

$= \frac{1}{4}\left(\frac{e^{-u}}{-1}\right) + C$

Let $u = x^4$

Then $du = 4x^3\,dx$

$= -\frac{1}{4}e^{-x^4} + C$

$\frac{1}{4}du = x^3\,dx$

b) $\int \frac{x^3 - x + 1}{x^2}\,dx \quad = \int (x^3 - x + 1)(x^{-2})\,dx$

$= \int \left(x - x^{-1} + x^{-2}\right)\,dx$

$= \frac{1}{2}x^2 - \ln|x| - x^{-1} + C$

7. Solve $\int x^3 \ln x\,dx$

Let $u = \ln x \qquad dv = x^3\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{4}x^4$

$\int u\,dv = uv - \int v\,du$

$\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^4\right)\left(\frac{1}{x}\,dx\right)$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$