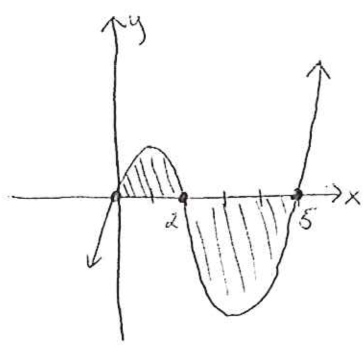
NAME KEY

Math 12
Test 4
Fall 2013

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by $y = x^3 - 7x^2 + 10x$ and the x-axis. Be sure to sketch a graph first!

$y = x(x^2 - 7x + 10)$
$y = x(x-2)(x-5)$



$$A = \int_0^2 [x^3 - 7x^2 + 10x - 0]\,dx + \int_2^5 [0 - (x^3 - 7x^2 + 10x)]\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2\right]_0^2 + \left[-\frac{1}{4}x^4 + \frac{7}{3}x^3 - 5x^2\right]_2^5$$

$$= \left[\left(4 - \frac{56}{3} + 20\right) - 0\right] + \left[\left(-\frac{625}{4} + \frac{875}{3} - 125\right) - \left(-4 + \frac{56}{3} - 20\right)\right]$$

$$= \frac{16}{3} + \left[\frac{-1875 + 3500 - 1500}{12} - \left(\frac{-16}{3}\right)\right]$$

$$= \frac{64}{12} + \frac{125}{12} + \frac{64}{12} = \frac{253}{12}$$

2. Find all four second-order partial derivatives of $f(x, y) = \ln(x^2 + y + 1)$. Do NOT simplify.

$$f_x = \frac{1}{x^2+y+1}(2x) = \frac{2x}{x^2+y+1}$$

$$f_y = \frac{1}{x^2+y+1}$$

$$f_{xx} = \frac{2(x^2+y+1) - (2x)(2x)}{(x^2+y+1)^2} \qquad f_{yx} = \frac{0 - 1(2x)}{(x^2+y+1)^2}$$

these match

$$f_{xy} = \frac{0 - (2x)(1)}{(x^2+y+1)^2} \qquad f_{yy} = \frac{0 - 1}{(x^2+y+1)^2}$$

3. Find and classify the critical points of $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

$f_x = -3x^2 + 4y = 0 \longrightarrow -3x^2 + 4x = 0$

$f_y = 4x - 4y = 0 \longrightarrow x = y$

$x(-3x+4) = 0$

$x = 0 \quad x = 4/3$

$y = 0 \quad y = 4/3$

Crit pts $(0,0), \left(\frac{4}{3}, \frac{4}{3}\right)$

$f_{xx} = -6x$

$f_{yy} = -4$

$f_{xy} = 4$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 24x - 16$

$D(0,0) = -16 < 0$, so $(0,0)$ gives a saddle point

$D\left(\frac{4}{3}, \frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 16 > 0$, $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -6\left(\frac{4}{3}\right) < 0$ max

$\left(\frac{4}{3}, \frac{4}{3}\right)$ gives a maximum.

4. Suppose two products have demand equations $D_1 = 2000 + \dfrac{100}{p_1 + 2} + 25p_2$ and $D_2 = 1500 - \dfrac{p_2}{p_1 + 7}$, where p_1 and p_2 are the respective prices of the products. Are the products competitive, complementary, or neither? Give an example of two products that might behave this way.

$$\frac{\partial D_1}{\partial p_2} = 25 > 0$$

$$\frac{\partial D_2}{\partial p_1} = -\left(\frac{0 - p_2(1)}{(p_1+7)^2}\right) = \frac{p_2}{(p_1+7)^2} > 0$$

Both positive, so products are competitive, like Coke & Pepsi.

5. A company manufactures a single product at two different locations. The cost of producing x_1 units at location 1 is $C_1 = 0.02x_1^2 + 4x_1 + 500$, and the cost of producing x_2 units at location 2 is $C_2 = 0.05x_2^2 + 4x_2 + 275$. The product sells for \$15 per unit. Find the quantity that should be produced at each location in order to maximize the total profit.

Profit = Revenue − Cost

$P = 15(x_1 + x_2) - C_1 - C_2$

$P = 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$

$\frac{\partial P}{\partial x_1} = 15 - 0.04x_1 - 4 = 0 \longrightarrow -0.04x_1 = -11 \quad x_1 = 275$

$\frac{\partial P}{\partial x_2} = 15 - 0.1x_2 - 4 = 0 \longrightarrow -0.1x_2 = -11 \quad x_2 = 110$

$\left.\begin{array}{l} \frac{\partial^2 P}{\partial x_2 \partial x_1} = 0 \\ \frac{\partial^2 P}{\partial x_1^2} = -0.04 \\ \frac{\partial^2 P}{\partial x_2^2} = -0.1 \end{array}\right\}$

$D = (-0.04)(-0.1) - 0 = 0.004$

$D(275, 110) = 0.004 > 0$

$\frac{\partial^2 P}{\partial x_1^2}(275, 110) < 0$, so max profit.

when $x_1 = 275$

$x_2 = 110$

6. Calculate $\int_4^\infty \frac{x}{\sqrt{(x^2+9)^3}}\,dx$.

$= \lim_{n\to\infty} \int_4^n \frac{x}{(x^2+9)^{3/2}}\,dx = \lim_{n\to\infty} \int_4^n x(x^2+9)^{-3/2}\,dx$

$u = x^2 + 9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \lim_{n\to\infty} \frac{1}{2}\int_{x=4}^{x=n} u^{-3/2}\,du$

$= \lim_{n\to\infty} \left[\frac{1}{2} \cdot -2\, u^{-1/2}\right]_{x=4}^{x=n}$

$= \lim_{n\to\infty} -(x^2+9)^{-1/2}\Big|_4^n$

$= \lim_{n\to\infty} -\left[\underbrace{(n^2+9)^{-1/2}}_{\to 0} - (16+9)^{-1/2}\right] = \frac{1}{5}$

7. The sales revenue of a company is found to be related to its advertising budget according to the formula $S = 20x + y^2 + 4xy$, where x is the amount spent on radio advertising and y is the amount spent on television advertising. If the company plans to spend \$30,000 on these two means of advertising, how should that budget be allocated between the two media in order to maximize sales revenue?

maximize $S = 20x + y^2 + 4xy$

subject to $x + y = 30$ (in thousands)

method (2)

$F(x,y,\lambda) = 20x + y^2 + 4xy - \lambda(x+y-30)$

$F_x = 20 + 4y - \lambda = 0 \longrightarrow 4y = \lambda - 20$

$y = \frac{1}{4}\lambda - 5$

$F_y = 2y + 4x - \lambda = 0 \longrightarrow 2(\frac{1}{4}\lambda - 5) + 4x - \lambda = 0$

$F_\lambda = -x - y + 30 = 0$

$\frac{1}{2}\lambda - 10 + 4x - \lambda = 0$

$4x = \lambda - \frac{1}{2}\lambda + 10 = \frac{1}{2}\lambda + 10$

$x = \frac{1}{8}\lambda + \frac{5}{2}$

method (1)

$\lambda = 20 + 4y$

$\lambda = 2y + 4x$

$20 + 4y = 2y + 4x$

$2y = 4x - 20$

$y = 2x - 10$

$-x - 2x + 10 + 30 = 0$

$40 = 3x$

$\frac{40}{3} = x$

$y = 2(\frac{40}{3}) - 10$

$= \frac{80}{3} - \frac{30}{3} = \frac{50}{3}$

$-\frac{1}{8}\lambda - \frac{5}{2} - \frac{1}{4}\lambda + 5 + 30 = 0$

$-\frac{3}{8}\lambda = -35 + \frac{5}{2}$

$-3\lambda = -280 + 20$

$3\lambda = 260$

$\lambda = \frac{260}{3} \rightarrow x = \frac{260}{24} + \frac{5}{2}$

$= \frac{65}{6} + \frac{15}{6} = \frac{80}{6}$

$= \frac{40}{3}$

$y = \frac{1}{4}(\frac{260}{3}) - 5$

$= \frac{65}{3} - \frac{15}{3} = \frac{50}{3}$

$x = \frac{40}{3} \longrightarrow$ spend \$13,333 on radio

$y = \frac{50}{3} \longrightarrow$ spend \$16,666 on TV