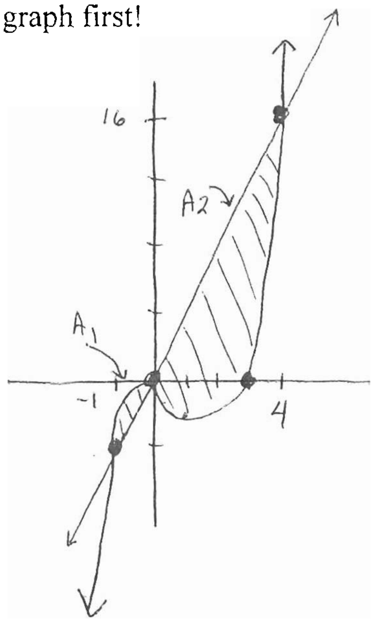
NAME Key

Math 12
Test 4
Spring 2011

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by $y = x^3 - 3x^2$ and $y = 4x$. Be sure to sketch a graph first!



Intersection points: $x^3 - 3x^2 = 4x$

$x^3 - 3x^2 - 4x = 0$

$x(x^2 - 3x - 4) = 0$

$x(x-4)(x+1) = 0$ $(0,0)$ $(4,16)$ $(-1,-4)$

$y = x^3 - 3x^2 = x^2(x-3)$
$(0,0)$ $(3,0)$

$$\text{Area} = A_1 + A_2 = \int_{-1}^{0} (x^3 - 3x^2 - 4x)\,dx + \int_0^4 (4x - x^3 + 3x^2)\,dx$$

$$= \left[\tfrac{1}{4}x^4 - x^3 - 2x^2\right]_{-1}^{0} + \left[2x^2 - \tfrac{1}{4}x^4 + x^3\right]_0^4$$

$$= \left[0 - \left(\tfrac{1}{4} + 1 - 2\right)\right] + \left[(32 - 64 + 64) - 0\right]$$

$$= \frac{3}{4} + 32 = \frac{131}{4}$$

2. Find the first-order partial derivatives of $f(x, y) = 5x\ln(x^2 + y)$. Do NOT simplify.

$$f_x = (5)(\ln(x^2+y)) + (5x)\left(\frac{1}{x^2+y}\right)(2x)$$

$$f_y = (5x)\left(\frac{1}{x^2+y}\right)$$

3. Find and classify the critical points of $f(x,y) = -2x^4 + 4xy - y^2 + 4x - 2y$

$f_x = -8x^3 + 4y + 4 = 0$

$f_y = 4x - 2y - 2 = 0$

$f_{xx} = -24x^2$

$f_{yy} = -2$

$f_{xy} = 4$

$-2x^3 + y + 1 = 0$

$2x - y - 1 = 0 \longrightarrow y = 2x - 1$

$-2x^3 + (2x-1) + 1 = 0$

$-2x^3 + 2x = 0$

$2x(-x^2+1) = 0$

$x = 0,\ y = -1, \quad x = 1,\ y = 1,$

$x = -1,\ y = -3$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$= 48x^2 - 16$

$D(0,-1) = -16 < 0 \longrightarrow (0,-1)$ gives a saddle point

$D(1,1) = 48 - 16 > 0,\ f_{xx}(1,1) = -24 < 0$

$D(-1,-3) = 48 - 16 > 0,\ f_{xx}(-1,-3) = -24 < 0$

$\Big\}$ $(1,1)$ and $(-1,-3)$ give maxima

4. For each three-dimensional surface below, determine the matching equation (a, b, c, d, or e).

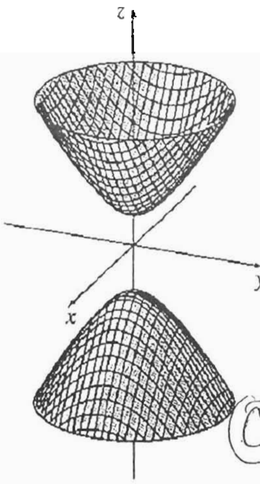
a) $z = x^2 + y^2$

b) $z = y^2 - x^2$

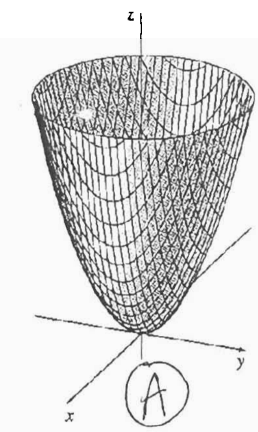
c) $z^2 - y^2 - x^2 = 1$

d) $\dfrac{x^2}{16} + \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1$

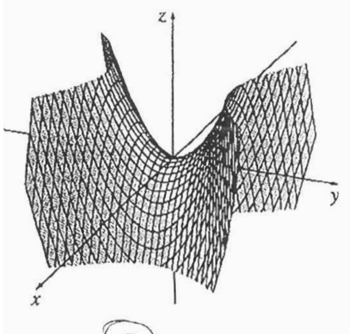
e) $z = 5(x^2 + y^2)^{-\frac{1}{2}}$



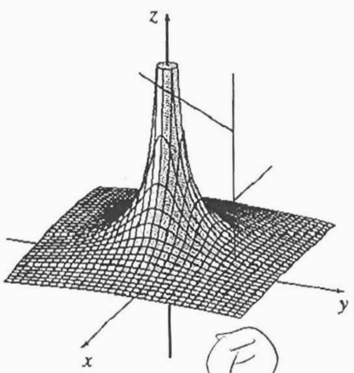
C



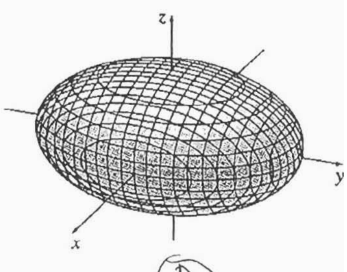
A



B

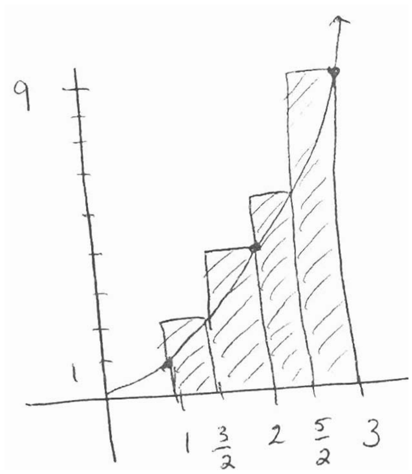


E



D

5. Using four rectangles, ***estimate*** the area under the curve $y = x^2$ between $x = 1$ and $x = 3$. Then find the ***exact*** area.



Right-Hand Endpoints

$$A \approx R_1 + R_2 + R_3 + R_4$$
$$\approx \tfrac{1}{2}\left(\tfrac{9}{4}\right) + \tfrac{1}{2}(4) + \tfrac{1}{2}\left(\tfrac{25}{4}\right) + \tfrac{1}{2}(9)$$
$$\approx \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{18}{8} \approx \frac{68}{8}$$
$$A \approx \frac{17}{2}$$

Left-Hand Endpoints

$$A \approx \tfrac{1}{2}(1) + \tfrac{1}{2}\left(\tfrac{9}{4}\right) + \tfrac{1}{2}(4) + \tfrac{1}{2}\left(\tfrac{25}{4}\right)$$
$$\approx \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} \approx \frac{54}{8} \approx \frac{27}{4}$$

Exact

$$A = \int_1^3 x^2\,dx = \tfrac{1}{3}x^3\Big|_1^3$$
$$= 9 - \tfrac{1}{3}$$
$$= \frac{26}{3}$$

6. Calculate $\int_1^{\infty} \frac{1}{x^2}\,dx$.

$$\int_1^{\infty} x^{-2}\,dx = \lim_{n\to\infty} \int_1^n x^{-2}\,dx = \lim_{n\to\infty} \left(-x^{-1}\right)\Big|_1^n$$
$$= \lim_{n\to\infty} \left(\frac{-1}{n} + \frac{1}{1}\right)$$
$$= 0 + 1$$
$$= 1$$

7. If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, the output at a factory will be $Q = 60x^{\frac{1}{3}}y^{\frac{2}{3}}$ units. If $120,000 is available how should this money be allocated between labor and equipment to generate the largest possible output?

$x + y = 120$ constraint

$Q = 60x^{1/3}y^{2/3}$ maximize

$F(x, y, \lambda) = 60x^{1/3}y^{2/3} - \lambda(x + y - 120)$

$F_x = 20x^{-2/3}y^{2/3} - \lambda = 0$

$F_y = 40x^{1/3}y^{-1/3} - \lambda = 0$

$F_\lambda = -x - y + 120 = 0$

$\lambda = 20x^{-2/3}y^{2/3} = 40x^{1/3}y^{-1/3}$

$20y = 40x$

$y = 2x$

$-x - 2x + 120 = 0$

$120 = 3x$

$x = 40$

$y = 80$

Spend \$40,000 on labor and \$80,000 on equipment