Mathematics 204
Spring 2014
Exam I

Your Printed Name: Solution.

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turn off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 7 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	6	7	**Sum**
Maximum points	16	16	14	14	9	14	17	100
Points earned								

1. [16] Find the general solution of

$$ty' = y + 2t^2 \sin(2t), \qquad t > 0.$$

1st order, linear ODE.

Write it into the standard form:

$$y' - \frac{1}{t} y = 2t\sin(2t) \qquad (*)$$

The integrating factor is

$$\mu = e^{-\int \frac{1}{t} dt} = e^{-\ln|t|} = e^{\ln\frac{1}{|t|}} = \frac{1}{t} \quad (\because t>0)$$

Multiplying μ at both sides of Eq. $(*)$, we have

$$\left[\frac{1}{t} y\right]' = 2\sin(2t)$$

Integrating it, we obtain

$$\frac{1}{t} y = -\cos(2t) + c$$

$$\Rightarrow \boxed{y = -t\cos(2t) + ct}$$

#.

2. [16] Solve the initial value problem

$$y' = te^{t-y}, \qquad y(0) = 0.$$

1st order, nonlinear ODE.

$$\frac{dy}{dt} = te^{t}e^{-y}$$

$$\Rightarrow \quad e^{y}dy = te^{t} \qquad (*)$$

Integrating Eq. (*) at both sides, we have

$$e^{y} = \int te^{t} = te^{t} - e^{t} + c.$$

Initial condition:

$$e^{0} = 0 - 1 + c \Rightarrow c = 1 + 1 = 2$$

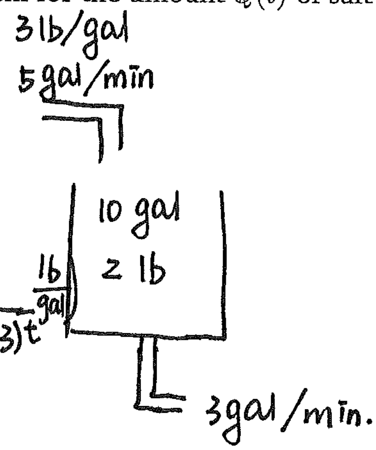
the solution is:

$$\boxed{e^{y} = te^{t} - e^{t} + 2}$$

#

3. A tank initially contains 10 gallons of water in which 2 pounds of salt is dissolved. A mixture containing 3 pounds of salt per gallon of water is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at a rate of 3 gallon per minute.

(a) [11] Write, **BUT DO NOT SOLVE**, an initial value problem for the amount $Q(t)$ of salt in the tank at time t.



Rate of change = amount in − amount out

$$\frac{dQ}{dt} = \left(3\frac{\text{lb}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) - \left(3\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q(t)}{10+(5-3)t}\frac{\text{lb}}{\text{gal}}\right)$$

$$= 15 - \frac{3Q}{10+2t}$$

the initial value problem:

$$\boxed{\begin{cases} \dfrac{dQ}{dt} = 15 - \dfrac{3Q}{10+2t} \\ Q(0) = 2. \end{cases}}$$

#

(b) [3] If the volume of the tank is 200 gallons, find the time t when the tank becomes full.

At time t, the volume of the mixture in the tank is

$$V = 10 + (5-3)t = 10 + 2t.$$

when the tank becomes full, we have

$$V = 200 \Rightarrow \boxed{t = 95 \text{ min}}$$

#

4. [14] Consider the differential equation

$$\frac{dy}{dt} = y^2(9 - y^2)$$

(a) Find the equilibrium (or critical) points.

(b) Sketch the phase line (or phase portrait). Be sure to **SHOW YOUR WORK.**

(c) Classify each equilibrium point as asymptotically stable, unstable, or semi-stable.

(d) If $y = y(t)$ denotes the solution of the initial value problem

$$y' = y^2(9 - y^2), \qquad y(0) = 4,$$

find the limit $\lim_{t\to\infty} y(t)$.

(a) $\frac{dy}{dt} = 0 \Rightarrow y^2(9-y^2) = 0 \Rightarrow \boxed{y = 0,\ -3 \text{ or } 3.}$

(b)

y	y^2	$9-y^2$	y'
-4	+	-	-
-2	+	+	+
2	+	+	+
4	+	-	-

(c)

3 → Asymptotically stable

0 → Semi-stable

-3 → unstable.

(d) $\lim_{t\to\infty} y(t) = 3$, since $y = 3$ is an asymptotically stable equilibrium point.

5. [9] The Wronskian of $f(t)$ and $g(t)$ is t. If $f(t) = \frac{1}{2}$ and $g(0) = 1$, find $g(t)$.

The wronskian of f and g is

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & g \\ 0 & g' \end{vmatrix} = t$$

$\Rightarrow \frac{1}{2}g' - 0 \cdot g = t$

$\therefore \quad g' = 2t \Rightarrow g = t^2 + c.$

Notice $g(0) = 1.$

$\Rightarrow \quad 1 = 0 + c \Rightarrow c = 1$

The solution: $\boxed{g = t^2 + 1.}$

#

6. Find the general solution of the following differential equations:

(a) [7] $y'' + 2y' + 3y = 0$.

$r^2 + 2r + 3 = 0$

$\Rightarrow \quad r = \dfrac{-2 \pm \sqrt{4-12}}{2} = \dfrac{-2 \pm 2\sqrt{2}\,i}{2} = -1 \pm \sqrt{2}\,i$

$\Rightarrow \quad \boxed{y = c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t).}$

#

(b) [7] $y'' + 4y' + 4y = 0$.

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$\Rightarrow \quad r = -2$ repeated once.

$\boxed{y = c_1 e^{-2t} + c_2 t e^{-2t}}$

#

7. [16] Given that $y = t^2$ solves

$$t^2 y'' - 3ty' + 4y = 0, \qquad t > 0,$$

find a second linearly independent solution of the problem.

Assume a second solution $y_2 = t^2 v(t)$.

$\Rightarrow \quad y_2' = 2tv + t^2 v'$

$y_2'' = t^2 v'' + 4tv' + 2v$

Substituting y_2, y_2' and y_2'' into the ODE, we have

$$LHS = t^2 [t^2 v'' + 4tv' + 2v] - 3t [2tv + t^2 v'] + 4t^2 v$$

$$= t^4 v'' + (4t^3 - 3t^3) v' + (2t^2 - 6t^2 + 4t^2) v$$

$$= t^4 v'' + t^3 v' = RHS = 0$$

$$\Rightarrow \quad t^4 v'' + t^3 v' = 0$$

i.e. $tv'' + v' = 0 \quad (\because t > 0)$

Let $u = v'$, we have

$u' + \frac{1}{t} u = 0 \Rightarrow$ integrating factor $\mu = e^{\int \frac{1}{t} dt} = e^{\ln|t|}$

$= t \quad (\because t > 0)$

$\Rightarrow [tu]' = 0 \Rightarrow u = \frac{c_1}{t}$

That is, $v' = u = \frac{c_1}{t}. \quad (*)$

Integrating Eq. $(*)$ at both sides, we obtain $v = c_1 \ln t + c_2 \quad (\because t > 0)$

set $c_1 = 1$, $c_2 = 0$, we have $\boxed{y_2 = t^2 \ln t}$ $\quad W(y_1, y_2) = \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t\ln t + t \end{vmatrix} = t^3 \neq 0 \quad (\because t > 0)$

#.