Mathematics 204

Spring 2014

Exam III

Your Printed Name: Solution

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turn off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. **Express all solutions in real-valued, simplified form.**
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	**Sum**
Maximum points	20	20	20	20	20	100
Points earned						

1. (a) [17] Solve the initial value problem

$$y'' + 4y = \delta(t-\pi); \qquad y(0) = \frac{1}{2}, \quad y'(0) = 0.$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$\Rightarrow [s^2 Y(s) - s y(0) - \cancel{y'(0)}] + 4Y(s) = e^{-\pi s}$$

$$\Rightarrow [s^2+4]\, Y(s) = \frac{1}{2}s + e^{-\pi s}$$

$$\Rightarrow Y(s) = \frac{1}{2}\frac{s}{s^2+4} + \frac{e^{-\pi s}}{s^2+4}$$

$$= \frac{1}{2}\frac{s^2}{s^2+2^2} + \frac{1}{2}e^{-\pi s}\frac{2}{s^2+2^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\cos(2t) + \frac{1}{2}u_\pi(t)\sin(2(t-\pi))$$

$$= \frac{1}{2}\cos(2t) + \frac{1}{2}u_\pi(t)\sin(2t)$$

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(b) [3] Which is greater, $y(\pi)$ or $y(\frac{3\pi}{2})$? Justify your answer.

$$\because \quad y(\pi) = \frac{1}{2}\cos(2\pi) + \frac{1}{2}\sin(2\pi) = \frac{1}{2}$$

$$y(\tfrac{3\pi}{2}) = \frac{1}{2}\cos(3\pi) + \frac{1}{2}\sin(3\pi) = -\frac{1}{2}$$

$$\therefore \quad y(\pi) > y(\tfrac{3\pi}{2})$$

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2. [20] Solve the initial value problem

$$y'' + 4y' + 13y = h(t), \qquad y(0) = y'(0) = 0,$$

where the function

$$h(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 13, & \pi \le t < \infty \end{cases}$$

First, rewrite $h(t) = \mathcal{U}_\pi(t)\,[13-0] = 13\,\mathcal{U}_\pi(t)$.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = \mathcal{L}\{13\,\mathcal{U}_\pi(t)\}$$

$$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 13Y(s) = 13\frac{e^{-\pi s}}{s}$$

$$\Rightarrow (s^2+4s+13)Y(s) = 13\frac{e^{-\pi s}}{s}$$

$$\therefore\ Y(s) = 13\frac{e^{-\pi s}}{s(s^2+4s+13)} = e^{-\pi s}\frac{13}{s(s^2+4s+13)} = e^{-\pi s}H(s)$$

Here, we have denoted

$$H(s) = \frac{13}{s(s^2+4s+13)} = \frac{As+B}{s^2+4s+13} + \frac{C}{s}$$

$$= \frac{(A+C)s^2 + (B+4C)s + 13C}{s(s^2+4s+13)}$$

$$\therefore \begin{cases} A+C=0 & \Rightarrow A=-1 \\ B+4C=0 & \Rightarrow B=-4 \\ 13C=13 & \Rightarrow C=1 \end{cases}$$

$$\therefore\ H(s) = \frac{1}{s} - \frac{s+4}{s^2+4s+13}$$

$$= \frac{1}{s} - \left[\frac{s+4}{(s^2+4s+4)+9}\right]$$

cont'd. $$= \frac{1}{s} - \left[\frac{s+2}{(s+2)^2+3^2} + \frac{2}{3}\frac{3}{(s+2)^2+3^2}\right]$$

$$\Rightarrow h(t) = 1 - e^{-2t}\cos(3t) - \frac{2}{3}e^{-2t}\sin(3t)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{U}_\pi(t)\,h(t-\pi)$$

$$= \mathcal{U}_\pi(t)\Big[1 - e^{-2(t-\pi)}\cos(3(t-\pi)) - \frac{2}{3}e^{-2(t-\pi)}\sin(3(t-\pi))\Big]$$

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3. [20] Find the solution $y(t)$ of the integral equation

$$y(t) - 9\int_0^t (t-\tau)y(\tau)d\tau = 9t^2.$$

It can be rewritten as

$$y(t) - 9\,(t * y) = 9t^2$$

$$\Rightarrow \mathcal{L}\{y(t)\} - 9\mathcal{L}\{t * y\} = 9\mathcal{L}\{t^2\}$$

$$\Rightarrow Y(s) - 9\frac{1}{s^2}Y(s) = 9\frac{2}{s^3}$$

$$\Rightarrow Y(s)\left[1 - \frac{9}{s^2}\right] = \frac{18}{s^3}$$

$$\Rightarrow Y(s)\frac{s^2-9}{s^2} = \frac{18}{s^3} \Rightarrow Y(s) = \frac{18}{s(s^2-9)} = \frac{18}{s(s+3)(s-3)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-3}$$

$$= \frac{A(s^2-9) + Bs(s-3) + Cs(s+3)}{s(s+3)(s-3)}$$

$$= \frac{(A+B+C)s^2 + (3C-3B)s - 9A}{s(s+3)(s-3)}$$

$$\therefore \begin{cases} A+B+C=0 \\ 3C-3B=0 \\ -9A=18 \end{cases} \Rightarrow \begin{cases} B=1 \\ C=1 \\ A=-2 \end{cases}$$

$$\therefore Y(s) = -\frac{2}{s} + \frac{1}{s+3} + \frac{1}{s-3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = -2 + e^{-3t} + e^{3t}$$

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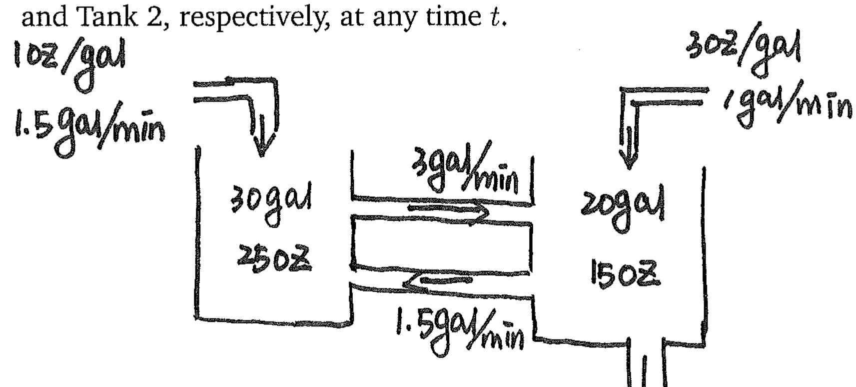
4. Consider a system of two interconnected tanks. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) [16] Set up, **BUT DO NOT SOLVE**, an initial value problem modeling the amount of salt in Tank 1 and Tank 2, respectively, at any time t.



Let $Q_1(t)$, $Q_2(t)$ denote the amount of salt in Tank 1 & Tank 2, respectively, at time t.

Rate of change = amount in − amount out.

$$\Rightarrow \quad \frac{dQ_1}{dt} = 1 \times 1.5 + 1.5\,\frac{Q_2}{20-t} - 3\,\frac{Q_1}{30}$$

$$\frac{dQ_2}{dt} = 3 \times 1 + 3\,\frac{Q_1}{30} - 5\,\frac{Q_2}{20-t}$$

$$\therefore \quad \frac{dQ_1}{dt} = 1.5 + 1.5\,\frac{Q_2}{20-t} - \frac{Q_1}{10}$$

$$\frac{dQ_2}{dt} = 3 + \frac{Q_1}{10} - 5\,\frac{Q_2}{20-t}$$

$Q_1(0) = 25$; $Q_2(0) = 15$

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(b) [4] For which times t is your model in part (a) valid? Explain why this is so.

The model is valid for $t \leq 20$ min. After $t = 20$ min, Tank 2 becomes empty.

5. [20] Find two linearly independent vector solutions of $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix},$$

and calculate their Wronskian.

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+4$$

$$= \lambda^2-\lambda-6+4 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)=0$$

$$\therefore \quad \lambda_1 = 2 \ ; \ \lambda_2 = -1$$

For $\lambda_1 = 2$, solve $(A-\lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 - v_2 = 0, \text{ i.e. } v_1 = v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

For $\lambda_2 = -1$, solve $(A-\lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 4v_1 - v_2 = 0, \text{ i.e. } v_2 = 4v_1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t}$$

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{vmatrix} = 4e^{t} - e^{t} = 3e^{t}$$

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