Mathematics 204

Fall 2010

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**

2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.

3. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transform formulas.

4. Once the exam begins, you will have 60 minutes to complete your solutions.

5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals, partial fraction decompositions, and matrix computations must be done by hand.

6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.

7. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	20	20	20	20	20	100

1.[20] Find the general solution of the system $\mathbf{x}' = \overbrace{\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}}^{A}\mathbf{x}$.

$\vec{x} = \vec{k}e^{\lambda t}$ leads to $\lambda\vec{k} = A\vec{k}$. The eigenvalues of A satisfy

$$0 = \det(A-\lambda I) = \begin{vmatrix}1-\lambda & 1\\ 4 & -2-\lambda\end{vmatrix} = (\lambda+2)(\lambda-1)-4 = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2).$$

Eigenvalues	Eigenvectors
$\lambda_1 = 2$	$\vec{k}^{(1)} = \begin{bmatrix}1\\1\end{bmatrix}$
$\lambda_2 = -3$	$\vec{k}^{(2)} = \begin{bmatrix}1\\-4\end{bmatrix}$

The eigenvectors of A satisfy $(A-\lambda I)\vec{k} = \vec{0}$

or $\begin{bmatrix}1-\lambda & 1\\ 4 & -2-\lambda\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

$\underline{\lambda = 2}$: $\begin{bmatrix}-1 & 1\\ 4 & -4\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Leftrightarrow \begin{cases}k_1 - k_2 = 0\\ \cancel{4k_1 - 4k_2 = 0}\end{cases}$ Redundant

$\therefore\ \vec{k}^{(1)} = \begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1\\k_1\end{bmatrix} = k_1\begin{bmatrix}1\\1\end{bmatrix}$. Take $k_1 = 1$ for convenience.

$\underline{\lambda = -3}$: $\begin{bmatrix}4 & 1\\ 4 & 1\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Leftrightarrow \begin{cases}4k_1 + k_2 = 0 & \text{so } k_2 = -4k_1\\ \cancel{4k_1 + k_2 = 0} & \text{Redundant}\end{cases}$

$\therefore\ \vec{k}^{(2)} = \begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1\\-4k_1\end{bmatrix} = k_1\begin{bmatrix}1\\-4\end{bmatrix}$. Take $k = 1$ for convenience.

Therefore solutions are $\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{\lambda_1 t} = \begin{bmatrix}1\\1\end{bmatrix}e^{2t}$ and $\vec{x}^{(2)}(t) = \vec{k}^{(2)}e^{\lambda_2 t} = \begin{bmatrix}1\\-4\end{bmatrix}e^{-3t}$.

$W(\vec{x}^{(1)}\ \vec{x}^{(2)})(t) = \det\begin{bmatrix}e^{2t} & e^{-3t}\\ e^{2t} & -4e^{-3t}\end{bmatrix} = -5e^{-t} \neq 0$. Therefore $\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)$ form a fundamental set of solutions for $\vec{x}' = A\vec{x}$. The general solution is

$$\vec{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t}$$

where c_1 and c_2 are arbitrary constants.

2.[20] Find the solution of the initial value problem.

$$x_1' = x_1 - x_2\,, \quad x_1(0) = 1$$
$$x_2' = x_1 + x_2\,, \quad x_2(0) = 2$$

A vector-matrix formulation of the system and initial conditions is $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$, $\vec{x}(0) =$

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $\vec{x} = \vec{k} e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda \vec{k} = A\vec{k}$.

The eigenvalues of A satisfy $0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-1)+1 = \lambda^2 - 2\lambda + 2$

$\therefore\ \lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$. Eigenvectors of A satisfy $(A - \lambda I)\vec{k} = \vec{0}$ so

$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. When $\lambda_1 = 1+i$ this becomes $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently

$\begin{cases} \cancel{-ik_1 - k_2} = 0 & \text{(Redundant; it is } -i \text{ times the second equation in the system.)} \\ k_1 - ik_2 = 0 & \end{cases}$

Thus, $k_1 = ik_2$, so an eigenvector corresponding to $\lambda = 1+i$ is

$\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} ik_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$. Taking $k_2 = 1$ yields $\vec{k}^{(1)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. A complex solution is

$\vec{x}^{(1)}(t) = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+i)t} = e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \left(\cos(t) + i\sin(t) \right)$. Real solutions are

$$\tilde{\vec{x}}^{(1)}(t) = \operatorname{Re}\left(\vec{x}^{(1)}(t)\right) = e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t) \right) = e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$\tilde{\vec{x}}^{(2)}(t) = \operatorname{Im}\left(\vec{x}^{(1)}(t)\right) = e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \right) = e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

$W(\tilde{\vec{x}}^{(1)}, \tilde{\vec{x}}^{(2)})(t) = \det \begin{bmatrix} -e^t \sin(t) & e^t \cos(t) \\ e^t \cos(t) & e^t \sin(t) \end{bmatrix} = -e^{2t} \neq 0$ so $\tilde{\vec{x}}^{(1)}(t), \tilde{\vec{x}}^{(2)}(t)$ forms a

fundamental set of solutions for $\vec{x}' = A\vec{x}$, and the general solution is $\vec{x}(t) = c_1 \tilde{\vec{x}}^{(1)}(t) + c_2 \tilde{\vec{x}}^{(2)}(t)$

We need to choose the arbitrary constants c_1 and c_2 so the initial condition is met:

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1 \tilde{\vec{x}}^{(1)}(0) + c_2 \tilde{\vec{x}}^{(2)}(0) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $c_1 = 2$ and $c_2 = 1$.

The solution of the initial value problem is

$$\vec{x}(t) = 2e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

3.[20] Given that $\Psi(t)=\begin{pmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix}$ is a fundamental matrix for $\mathbf{x}' = \overbrace{\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}}^{A}\mathbf{x}$, find the general solution of the nonhomogeneous system $\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{t}}_{\vec{g}(t)}$.

The general solution is $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t) = \Psi(t)\vec{c} + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds.$

$\Psi^{-1}(s) = \dfrac{1}{\det \Psi(s)}\begin{bmatrix} \psi_{22}(s) & -\psi_{12}(s) \\ -\psi_{21}(s) & \psi_{11}(s) \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} e^{-s} & -e^{-s} \\ -e^{s} & 3e^{s} \end{bmatrix}$. A particular solution is

$$\vec{x}_p(t) = \Psi(t)\int_{t_0}^{t} \frac{1}{2}\begin{bmatrix} e^{-s} & -e^{-s} \\ -e^{s} & 3e^{s} \end{bmatrix}\begin{bmatrix} 3e^{s} \\ e^{s} \end{bmatrix}ds = \Psi(t)\int_{t_0}^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}ds = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} t \\ 0 \end{bmatrix}.$$ (We've set $t_0 = 0$.)

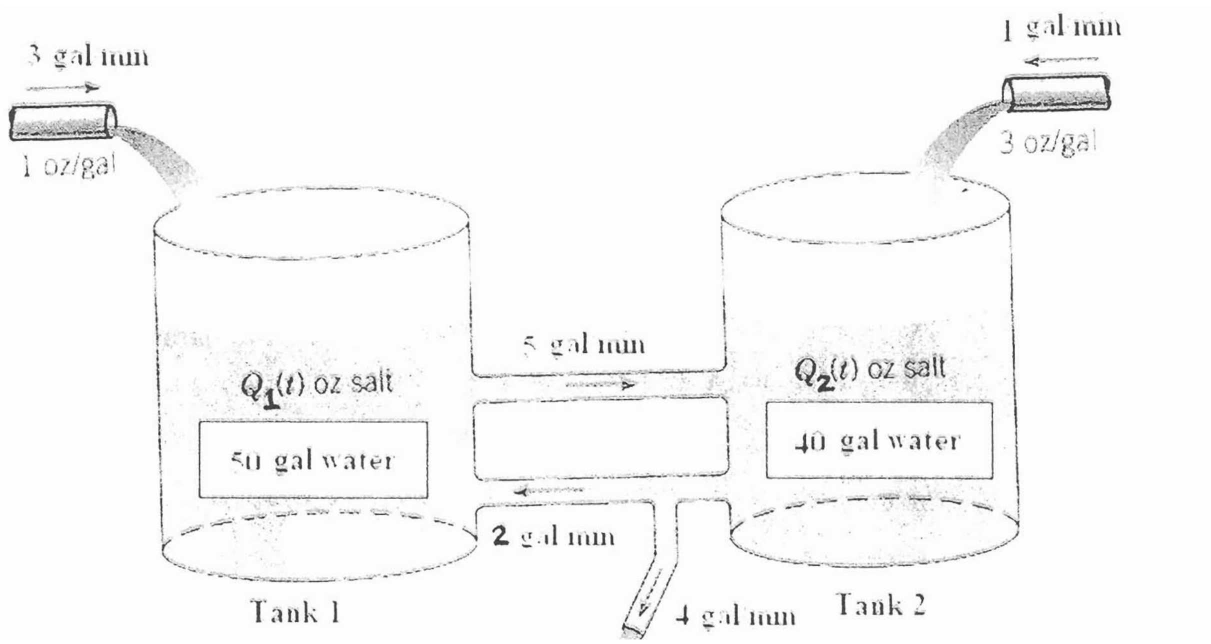
The general solution is

$$\vec{x}(t) = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} t \\ 0 \end{bmatrix}$$

or

$$\vec{x}(t) = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}te^{t}.$$

4.[20] Consider two interconnected tanks as shown below. Initially 40 oz of salt is in tank 1 and 50 oz of salt is in tank 2. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt, in oz, in each tank at time t. Assume the solution in the tanks is fully mixed. Write down, **BUT DO NOT SOLVE**, a system of differential equations and initial conditions that model the flow process.



We apply Net Rate = Rate In − Rate Out to each tank.

Tank 1: $$\frac{dQ_1}{dt} = \left(3\,\frac{gal}{min}\right)\left(1\,\frac{oz}{gal}\right) + \left(2\,\frac{gal}{min}\right)\left(\frac{Q_2\,oz}{40\,gal}\right) - \left(\frac{5\,gal}{min}\right)\left(\frac{Q_1\,oz}{50\,gal}\right)$$

Tank 2: $$\frac{dQ_2}{dt} = \left(1\,\frac{gal}{min}\right)\left(\frac{3\,oz}{gal}\right) + \left(\frac{5\,gal}{min}\right)\left(\frac{Q_1\,oz}{50\,gal}\right) - \left(\frac{6\,gal}{min}\right)\left(\frac{Q_2\,oz}{40\,gal}\right)$$

Simplifying and supplementing with the initial values of salt gives

$$\begin{cases} Q_1' = -0.1\,Q_1 + 0.05\,Q_2 + 3, & Q_1(0) = 40 \\ Q_2' = 0.1\,Q_1 - 0.15\,Q_2 + 3, & Q_2(0) = 50 \end{cases}$$

5.[20] Find the solution of the initial value problem $y'' + 2y' + 2y = \delta(t-5)$, $y(0) = 0$, $y'(0) = 0$. Then compute the value of the solution, accurate to five decimal places, when $t = 6$.

We use the Laplace transform method.

$$\mathcal{L}\{y'' + 2y' + 2y\}(s) = \mathcal{L}\{\delta(t-5)\}(s)$$

#8,9,10 on the Laplace transform table

$$s^2 Y(s) - s\overset{0}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + 2\left(sY(s) - \overset{0}{\cancel{y(0)}}\right) + 2Y(s) \doteq e^{-5s}$$

$$\left(Y(s) = \mathcal{L}\{y\}(s)\right)$$

$$(s^2 + 2s + 2)Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{s^2 + 2s + 2}$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\left\{ e^{-5s} \cdot \frac{1}{s^2 + 2s + 2} \right\}$$

#12 on Laplace transform table

$$y(t) \doteq u_5(t) f(t-5)$$

#5 on Laplace transform table

$$\text{where } f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+2s+2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{\underbrace{s^2+2s+1+1}_{\text{Completing the square}}} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+1} \right\} \doteq e^{-t}\sin(t).$$

$$\text{Therefore } \boxed{y(t) = u_5(t) e^{-(t-5)} \sin(t-5)}.$$

$$y(6) = \underbrace{u_5(6)}_{1} e^{-(6-5)} \sin(6-5) = e^{-1}\sin(1) \doteq \boxed{0.30956}$$

(Note: A common error here is to forget your calculator must be in radian mode.)

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. e^{at}	$\dfrac{1}{s-a}$
3. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3....$
4. $e^{at}t$	$\dfrac{1}{(s-a)^2}$
5. $e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
6. $e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
7. $e^{at}f(t)$	$F(s-a)$
8. $f'(t)$	$sF(s)-f(0)$
9. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$
10. $\delta(t-a)$	e^{-as}
11. $u_a(t)$	$\dfrac{e^{-as}}{s}$
12. $u_a(t)f(t-a)$	$e^{-as}F(s)$
13. $\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$

2010 Fall Semester, Math 204 Hour Exam III, Master List

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Section	Instructor	No. Taking Exam III
A	Wintz	35
B	Wintz	33
C	Wintz	33
D	Van Rhein	38
E	Heim	42
F	Grow	34
G	Fitch	38
H	Fitch	30
J	He	25
K	Heim	34
L	Singler	29

Number taking exam: 371
Median: 81
Mean: 77.63
Standard Deviation: 18.52

Number receiving A's: 122 32.9%
Number receiving B's: 74 19.9
Number receiving C's: 66 17.8
Number receiving D's: 55 14.8
Number receiving F's: 54 14.6
29.4