Mathematics 204

Spring 2012

Final Exam

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not** allowed to use a **calculator** on this exam.
4. The final exam consists of this cover page, 10 pages of problems containing 10 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.
9. If you work a problem in more than one way, clearly indicate which solution you want us to grade, preferably by crossing out the others. If you use more than one method, one of which is wrong and is not crossed out, you will not receive full credit on that problem.

problem	1	2	3	4	5	6	7	8	9	10	**Sum**
points earned											
maximum points	20	20	20	20	20	20	20	20	20	20	200

1.[20] Find the explicit solution of the initial value problem $y' = t(y^2 - 1)$, $y(0) = 3$.

This equation has the form $y' = g(t)h(y)$ so it is first order separable.
Write $\frac{dy}{dt} = t(y^2-1)$ and rearrange to obtain

$$\frac{dy}{y^2-1} = t\,dt.$$

Now integrate both sides: $\int \frac{1}{y^2-1}\,dy = \int t\,dt = \frac{t^2}{2} + c_1$.

The integrand in the left member can be decomposed into partial fractions as

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}. \qquad (A, B \text{ constants})$$

Multiplying through by $(y-1)(y+1)$ gives $1 = A(y+1) + B(y-1)$.

To find A, set $y = 1$: $1 = A(1+1) + B(1-1)$ so $A = \frac{1}{2}$.

To find B, set $y = -1$: $1 = A(-1+1) + B(-1-1)$ so $B = -\frac{1}{2}$.

Therefore

$$\frac{t^2}{2} + c_1 = \int \left(\frac{1/2}{y-1} + \frac{-1/2}{y+1} \right) dy = \frac{1}{2}\int \frac{dy}{y-1} - \frac{1}{2}\int \frac{dy}{y+1}$$

$$\frac{t^2}{2} + c_1 = \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|. \quad \text{(Implicit General Solution)}$$

Multiplying through by 2 and exponentiating both members gives

$$e^{t^2+c} = \left|\frac{y-1}{y+1}\right| \quad \text{or} \quad Ke^{t^2} = \frac{y-1}{y+1} \quad \text{where } K = \pm e^{c} = \pm e^{2c_1}.$$

Solving for y we obtain $Ke^{t^2}(y+1) = y-1$ or equivalently

$$Ke^{t^2}y + Ke^{t^2} = y - 1 \Rightarrow Ke^{t^2} + 1 = y(1 - Ke^{t^2})$$

$$\Rightarrow y(t) = \frac{1 + Ke^{t^2}}{1 - Ke^{t^2}} \quad \text{(Explicit General Solution)}.$$

(OVER)

When $t=0$ we want to choose K such that $y=3$:

$$3 = y(0) = \frac{1+Ke^0}{1-Ke^0} = \frac{1+K}{1-K} \Rightarrow 3(1-K) = 1+K$$

$$\Rightarrow 3-3K = 1+K \Rightarrow 2 = 4K \Rightarrow K = \tfrac{1}{2}.$$

Thus $\boxed{y(t) = \dfrac{1+\frac{1}{2}e^{t^2}}{1-\frac{1}{2}e^{t^2}}}$ or equivalently $\boxed{y(t) = \dfrac{2+e^{t^2}}{2-e^{t^2}}}$

2.[20] A 100 gallon tank originally contains 20 gallons of water and 5 pounds of salt. Then water containing 1/2 pound of salt per gallon is poured into the tank at a rate of 3 gallons per minute, and the well-stirred mixture leaves at a rate of 2 gallons per minute. Set up, **BUT DO NOT SOLVE**, an initial value problem that models the amount of salt in the tank for any time in the interval $0 \le t \le 80$.

Let $A(t)$ denote the number of pounds of salt in the tank at time t minutes. We use

net rate of change of A = rate of inflow of A − rate of outflow of A.

Symbolically this is

$$\frac{dA}{dt} = \left(\frac{3\text{ gal}}{\text{min}}\right)\left(\frac{\tfrac{1}{2}\text{ pound}}{\text{gal}}\right) - \left(\frac{2\text{ gal}}{\text{min}}\right)\left(\frac{A(t)\text{ pounds}}{V(t)\text{ gal}}\right)$$

where $V(t)$ = volume of solution in the tank at time t minutes

$= 20 + t$ gallons.

Note that $A(0) = 5$ since the tank initially contained 5 pounds of salt.

Thus

$$\boxed{\frac{dA}{dt} = \frac{3}{2} - \frac{2A}{20+t}, \quad A(0) = 5}$$

(A in pounds, t in minutes)

is an initial value problem that models the amount of salt in the tank for $0 \le t \le 80$ minutes.

<u>Note:</u> After 80 minutes the tank overflows ($20 + 80 = 100$ gallons) so the model is no longer valid.

3.[20] If the Wronskian of the functions f and g is -3 and if $f(t)=t^2$, find $g(t)$.

$$W(f,g)(t) = -3 \quad \text{means} \quad \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = -3$$

so substituting $f(t)=t^2$ gives $\begin{vmatrix} t^2 & g(t) \\ 2t & g'(t) \end{vmatrix} = -3$. Equivalently

$$t^2 g'(t) - 2t\, g(t) = -3 .$$

This equation is first order linear. Normalizing yields

$$(*) \qquad g'(t) - \frac{2}{t} g(t) = -\frac{3}{t^2} .$$

An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{2}{t} dt} = e^{-2\ln(t) + \cancel{C}^{\,0}} = e^{\ln(t^{-2})}$ $= t^{-2}$. Multiplying $(*)$ by the integrating factor yields

$$\underbrace{t^{-2} g'(t) - 2t^{-3} g(t)}_{\text{Exact!}} = -3t^{-4} .$$

$$\frac{d}{dt}\left[t^{-2} g(t) \right] \overset{!}{=} -3t^{-4}$$

Check: $(t^{-2}g(t))' \overset{\checkmark}{=} t^{-2}g'(t) + -2t^{-3}g(t)$ by the product rule $(fg)' = fg' + f'g$.

Integrating both sides gives $t^{-2}g(t) = \int -3t^{-4}dt = -3\left(\frac{t^{-3}}{-3}\right) + c$

or $g(t) = t^2\left(t^{-3} + c\right)$

$$\boxed{g(t) = \frac{1}{t} + ct^2}$$

where c is an arbitrary constant.

4.[20] Solve $4y'' + y = 2\sec(t/2)$ on the interval $-\pi < t < \pi$.

Normalizing, $y'' + \frac{1}{4}y = \frac{1}{2}\sec(t/2)$. This is a second order linear nonhomogeneous equation with general solution $y = y_c + y_p$ where y_c is the general solution of the associated homogeneous equation $y'' + \frac{1}{4}y = 0$ and y_p is any particular solution of the nonhomogeneous equation.

$y = e^{rt}$ in $y'' + \frac{1}{4}y = 0$ leads to $r^2 + \frac{1}{4} = 0$ so $r = \pm \frac{i}{2}$. Therefore $y_1(t) = \cos(t/2)$, $y_2(t) = \sin(t/2)$ is a fundamental set of solutions to $y'' + \frac{1}{4}y = 0$ since $W(y_1, y_2)(t) = \begin{vmatrix} \cos(t/2) & \sin(t/2) \\ -\frac{1}{2}\sin(t/2) & \frac{1}{2}\cos(t/2) \end{vmatrix} = \frac{1}{2} \neq 0$.

Consequently, $y_c(t) = c_1\cos(t/2) + c_2\sin(t/2)$ for arbitrary constants c_1, c_2.

We use variation of parameters to find a particular solution to the nonhomogeneous equation: $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = \int \frac{-y_2(t)g(t)}{W(t)}dt = \int \frac{-\sin(t/2)\cdot\frac{1}{2}\sec(t/2)}{\frac{1}{2}}dt = -\int \tan(t/2)dt$$

$$= 2\ln|\cos(t/2)| + \cancelto{0}{c_1}$$

and

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}dt = \int \frac{\cos(t/2)\cdot\frac{1}{2}\sec(t/2)}{\frac{1}{2}}dt = \int 1\,dt = t + \cancelto{0}{c_2}.$$

Consequently $y_p(t) = 2\cos(t/2)\ln|\cos(t/2)| + t\sin(t/2)$. Thus on $-\pi < t < \pi$

$$\boxed{y(t) = c_1\cos(t/2) + c_2\sin(t/2) + 2\cos(t/2)\ln(\cos(t/2)) + t\sin(t/2)}$$

is the general solution of the nonhomogeneous equation.

5.[20] Find the general solution of the differential equation $y^{(4)} - 2y'' + y = e^{2t}$.

This is a fourth order linear nonhomogeneous equation so the general solution is $y = y_c + y_p$ where y_c is the general solution of the associated homogeneous equation and y_p is any particular solution to the nonhomogeneous equation.

$y = e^{rt}$ in $y^{(4)} - 2y'' + y = 0$ leads to $r^4 - 2r^2 + 1 = 0$. This factors as $(r^2 - 1)^2 = 0$ or $(r-1)^2(r+1)^2 = 0$. Therefore $r = 1$ (multiplicity 2) and $r = -1$ (multiplicity 2) are the roots of the characteristic equation so $y_c = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t}$ is the general solution of the associated homogeneous equation where c_1, c_2, c_3, c_4 are arbitrary constants.

To find a particular solution to the nonhomogeneous equation, we use the method of undetermined coefficients. A trial form for a particular solution is $y_p(t) = Ae^{2t}$ where A is a constant to be determined such that y_p solves the nonhomogeneous equation; i.e. so that

$$y_p^{(4)} - 2y_p'' + y_p = e^{2t}.$$

Substituting for y_p and its derivatives gives

$$2^4 Ae^{2t} - 2(2)^2 Ae^{2t} + Ae^{2t} = e^{2t}.$$

Canceling e^{2t} and simplifying gives $9A = 1$ so $A = 1/9$. Thus

$$\boxed{y(t) = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t} + \frac{1}{9}e^{2t}}$$

is the general solution of the nonhomogeneous equation.

6.[20] (a) Solve the initial value problem $y''+9y=3\delta(t-3\pi)-3\delta(t-5\pi)$, $y(0)=0$, $y'(0)=0$.
(b) Write your solution as a piecewise defined function and sketch its graph on the interval $0\le t\le 7\pi$.

(a) We use the Laplace transform method. Let $y=y(t)$ solve the IVP. Then

$$y''(t)+9y(t)=3\,\delta(t-3\pi)-3\,\delta(t-5\pi) \qquad (t\geq 0)$$

so taking the Laplace transform of both sides gives

$$\mathcal{L}\{y''+9y\}(s)=\mathcal{L}\{3\,\delta(t-\pi)-3\,\delta(t-5\pi)\}(s)$$

or

$$s^2\mathcal{L}\{y\}(s)-s\,y(0)-y'(0)+9\mathcal{L}\{y\}(s)=3e^{-\pi s}-3e^{-5\pi s}$$

by linearity and entries 6 and 9 in the short table of Laplace transforms. Substituting $y(0)=0=y'(0)$ and solving for $\mathcal{L}\{y\}(s)$ we find

$$\mathcal{L}\{y\}(s)=\frac{3e^{-\pi s}}{s^2+9}-\frac{3e^{-5\pi s}}{s^2+9}.$$

Thus

$$y(t)=\mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{s^2+9}-\frac{3e^{-5\pi s}}{s^2+9}\right\}$$

$$=u_\pi(t)f(t-\pi)-u_{5\pi}(t)f(t-5\pi)$$

by entry 8 in the Laplace transform table where

$$f(t)=\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}=\sin(3t);$$

here we used entry 3 in the Laplace transform table. Consequently

$$\boxed{y(t)=u_\pi(t)\sin(3(t-\pi))-u_{5\pi}(t)\sin(3(t-5\pi))}$$

solves the IVP. (OVER)

(b) Note that $\sin(3(t-\pi)) = \sin(3t-3\pi)$

$$= \sin(3t)\underbrace{\cos(3\pi)}_{-1} - \cos(3t)\underbrace{\sin(3\pi)}_{0}$$

$$= -\sin(3t)$$

and similarly $\sin(3(t-5\pi)) = -\sin(3t)$. Since $u_c(t) = \begin{cases} 0 & \text{if } t<c, \\ 1 & \text{if } t \geq c, \end{cases}$

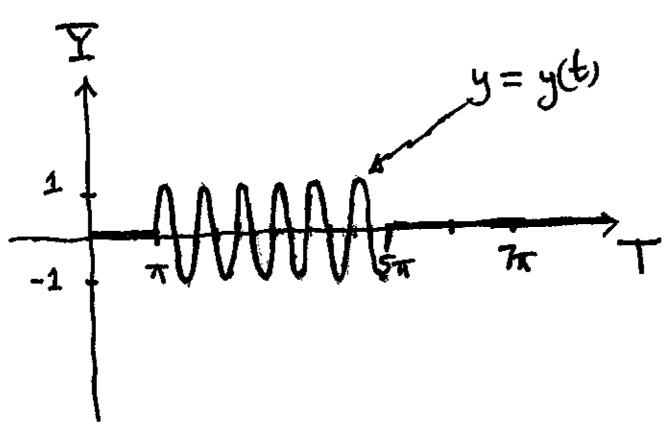
we can rewrite the solution in (a) as

$$y(t) = \begin{cases} 0 & \text{if } t < \pi, \\ -\sin(3t) & \text{if } \pi \leq t < 5\pi, \\ -\sin(3t) - (-\sin(3t)) & \text{if } 5\pi \leq t, \end{cases}$$

or

$$\boxed{y(t) = \begin{cases} 0 & \text{if } t < \pi, \\ -\sin(3t) & \text{if } \pi \leq t < 5\pi, \\ 0 & \text{if } 5\pi \leq t. \end{cases}}$$

The graph of the solution on $0 \leq t \leq 7\pi$ follows.



7.[20] Solve the integro-differential equation $y''(t)-\int_0^t \tau y(t-\tau)\,d\tau = 1$ subject to the initial conditions $y(0)=0,\ y'(0)=0.$

Using the definition of the convolution product,

$$f*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau,$$

we see that the integro-differential equation can be written as

$$(\dagger) \qquad y'' - f*y = 1$$

where $f(t)=t$. Taking the Laplace transform of both sides of $(\dagger)$ we have

$$(\dagger\dagger) \qquad s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) - \mathcal{L}\{f\}(s)\mathcal{L}\{y\}(s) = \mathcal{L}\{1\}(s)$$

by linearity and entries 6 and 5 in the Laplace transform table. But $\mathcal{L}\{f\}(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2}$ and $\mathcal{L}\{1\}(s) = \frac{1}{s}$ by entry 2 in the Laplace transform table. Substituting these expressions and the initial conditions $y(0)=0=y'(0)$ in $(\dagger\dagger)$ gives

$$s^2\mathcal{L}\{y\}(s) - \frac{1}{s^2}\mathcal{L}\{y\}(s) = \frac{1}{s}.$$

Rearranging gives

$$\left(\frac{s^4-1}{s^2}\right)\mathcal{L}\{y\}(s) = \frac{1}{s}$$

or

$$\mathcal{L}\{y\}(s) = \left(\frac{s^2}{s^4-1}\right)\left(\frac{1}{s}\right) = \frac{s}{s^4-1}.$$

(OVER)

Writing

$$\frac{s}{s^4-1} = \frac{s}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1},$$

we have upon multiplying through by $(s^2+1)(s^2-1)$ that

$$s = (As+B)(s^2-1) + C(s+1)(s^2+1) + D(s-1)(s^2+1).$$

To find C, set $s=1$: $\quad 1 = 0 + 4C + 0 \quad$ so $\quad C = 1/4$.

To find D, set $s=-1$: $\quad -1 = 0 + 0 - 4D \quad$ so $\quad D = 1/4$.

To find A and B set $s=i$: $\quad i = (Ai+B)(-2) + 0 + 0$.

Rewriting the last equation in the form $0 + 1i = -2B - 2Ai$ we see that $B=0$ and $A = -1/2$. Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^4-1}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s}{s^2+1} + \frac{1/4}{s-1} + \frac{1/4}{s+1}\right\}$$

so

$$\boxed{y(t) = \frac{1}{4}e^{t} + \frac{1}{4}e^{-t} - \frac{1}{2}\cos(t)}$$

by entries 1 and 4 in the Laplace transform table.

8.[20] Find a fundamental matrix for the system $\mathbf{x}' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -2 & 2 \end{pmatrix} \mathbf{x}$. Write $\vec{x}' = A\vec{x}$.

We assume solutions of $\vec{x}' = A\vec{x}$ of the form $\vec{x} = \vec{k}e^{\lambda t}$ where λ and $\vec{k}$ are constants. Then $\vec{x}' = \lambda \vec{k} e^{\lambda t}$ so substituting in $\vec{x}' = A\vec{x}$ gives $\lambda \vec{k} e^{\lambda t} = A\vec{k}e^{\lambda t}$. Canceling $e^{\lambda t}$ gives the eigenvalue equation for A: $\lambda \vec{k} = A\vec{k}$. Therefore λ satisfies $0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -3 \\ 0 & -2 & 2-\lambda \end{vmatrix}$.

Expanding the determinant gives

$$0 = (2-\lambda)\begin{vmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{vmatrix} = (2-\lambda)\big((1-\lambda)(2-\lambda) - 6\big) = (2-\lambda)(\lambda^2 - 3\lambda - 4)$$

or $\quad 0 = (2-\lambda)(\lambda-4)(\lambda+1)$. Hence $\lambda = 2$, $\lambda = 4$, or $\lambda = -1$.

Eigenvectors corresponding to $\lambda = 2$ satisfy $(A - \lambda I)\vec{k} = \vec{0}$ so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \Leftrightarrow \begin{cases} 0 = 0 \\ 0 = -k_2 - 3k_3 \\ 0 = -2k_2 \end{cases} \quad \text{so} \quad k_2 = k_3 = 0$$

(and k_1 arbitrary)

Thus $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 2$. Hence

$\vec{x}^{(1)} = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$ solves $\vec{x}' = A\vec{x}$.

Eigenvectors corresponding to $\lambda = 4$ satisfy $(A - \lambda I)\vec{k} = \vec{0}$ so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -2 & -2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \Leftrightarrow \begin{cases} 0 = -2k_1 \\ 0 = -3k_2 - 3k_3 \\ 0 = -2k_2 - 2k_3 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = -k_3 \end{cases}.$$

Thus $\vec{k}^{(2)} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

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Therefore $\vec{x}^{(2)} = \vec{k}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{4t}$ solves $\vec{x}' = A\vec{x}$.

Eigenvectors corresponding to $\lambda = -1$ satisfy $(A - \lambda I)\vec{k} = \vec{0}$ so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \Leftrightarrow \begin{cases} 0 = 3k_1 \\ 0 = 2k_2 - 3k_3 \\ 0 = -2k_2 + 3k_3 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = \frac{3}{2} k_3 \end{cases}.$$

Hence $\vec{k}^{(3)} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2}k_3 \\ k_3 \end{bmatrix} = \frac{k_3}{2} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -1$. Therefore $\vec{x}^{(3)} = \vec{k}^{(3)} e^{\lambda_3 t} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} e^{-t}$ solves $\vec{x}' = A\vec{x}$.

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{3}) = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & -e^{4t} & 3e^{-t} \\ 0 & e^{4t} & 2e^{-t} \end{vmatrix} = e^{2t} \cdot e^{4t} \cdot e^{-t} \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= e^{5t}(1) \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} = -5e^{5t} \neq 0.$$ Therefore

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{4t}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} e^{-t}}$$

is a fundamental set of solutions to $\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix} \vec{x}$.

9.[20] Find the general solution of the system

$$\frac{dx}{dt} = -3x + y$$

$$\frac{dy}{dt} = -x - y.$$

Describe how the solutions behave as $t \to \infty$.

Write $\vec{x}' = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$. We assume solutions of the form $\vec{x} = \vec{k}e^{\lambda t}$ where λ and $\vec{k}$ are constants. Then $\vec{x}' = \lambda \vec{k} e^{\lambda t}$ so substituting in $\vec{x}' = A\vec{x}$ gives $\lambda \vec{k} e^{\lambda t} = A\vec{k}e^{\lambda t}$. Canceling $e^{\lambda t}$ from both sides of the equation leads to the eigenvalue equation for A: $\lambda\vec{k} = A\vec{k}$. The eigenvalues λ satisfy $0 = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 1$ so $0 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$. Therefore $\lambda = -2$ is an eigenvalue of A with multiplicity two. An eigenvector of A corresponding to $\lambda = -2$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$, i.e. $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -k_1 + k_2 = 0 \\ -k_1 + k_2 = 0 \end{cases}$ so $k_2 = k_1$. Thus $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = -2$. Consequently $\vec{x}^{(1)} = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ solves $\vec{x}' = A\vec{x}$.

To get a second linearly independent solution, we assume a solution of the form $\vec{x}(t) = t\vec{k}e^{\lambda t} + \vec{\ell}e^{\lambda t}$ where λ, $\vec{k}$, and $\vec{\ell}$ are constants. Differentiating gives $\vec{x}'(t) = \vec{k}e^{\lambda t} + \lambda t \vec{k} e^{\lambda t} + \lambda \vec{\ell} e^{\lambda t}$ so substituting in $\vec{x}' = A\vec{x}$ and simplifying leads to the system $\begin{cases} (A - \lambda I)\vec{k} = \vec{0}, \\ (A - \lambda I)\vec{\ell} = \vec{k}. \end{cases}$ We have already solved the first equation in the system: $\lambda = -2$ and $\vec{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (up to a constant factor). Substituting these values in the second equation in the system

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leads to $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \begin{cases} -\ell_1 + \ell_2 = 1 \\ -\ell_1 + \ell_2 = 1 \end{cases} \Rightarrow \ell_2 = 1 + \ell_1.$

Therefore $\vec{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_1 + 1 \end{bmatrix} = \ell_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We may take $\ell_1 = 0$

to get $\vec{\ell} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore

$$\vec{x}^{(2)}(t) = t\vec{k}e^{\lambda t} + \vec{\ell}e^{\lambda t} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-2t}$$

is a second linearly independent solution of $\vec{x}' = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}\vec{x}$. The general solution is

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-2t} \right)}$$

where c_1 and c_2 are arbitrary constants.

As $t \to \infty$, $e^{-2t} \to 0$ so $\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)e^{-2t} \to \boxed{\vec{0}}$

for all constants c_1 and c_2

10.[20] Solve the initial value problem

$$x'' = -5x + 4y, \quad x(0) = 4,\ x'(0) = 8,$$
$$y'' = x - 8y, \quad y(0) = 1,\ y'(0) = 2.$$

Suggestion: Although the matrix method can be used to successfully solve this problem, there are better methods.

Laplace Transform Method: Suppose a solution to the system is $x = x(t)$, $y = y(t)$.

Then

$$\begin{cases} x''(t) = -5x(t) + 4y(t) \\ y''(t) = x(t) - 8y(t) \end{cases}$$

for all $t \geq 0$. Taking the Laplace transform of both equations gives

$$\begin{cases} s^2\mathcal{L}\{x\}(s) - s x(0) - x'(0) = -5\mathcal{L}\{x\}(s) + 4\mathcal{L}\{y\}(s) \\ s^2\mathcal{L}\{y\}(s) - s y(0) - y'(0) = \mathcal{L}\{x\}(s) - 8\mathcal{L}\{y\}(s) \end{cases}$$

by linearity and entry 6 in the Laplace transform table. Applying the initial conditions and rearranging terms yields

$$\begin{cases} (s^2+5)\mathcal{L}\{x\}(s) - 4\mathcal{L}\{y\}(s) = 4s + 8 \\ -\mathcal{L}\{x\}(s) + (s^2+8)\mathcal{L}\{y\}(s) = s + 2. \end{cases}$$

Multiplying the second equation in this system by s^2+5 and adding the result to the first equation eliminates $\mathcal{L}\{x\}(s)$ producing

$$\left[(s^2+8)(s^2+5) - 4\right]\mathcal{L}\{y\}(s) = (s+2)(s^2+5) + 4s + 8.$$

Simplifying and solving for $\mathcal{L}\{y\}(s)$, we have

(OVER)

$$\mathcal{L}\{y\}(s) = \frac{(s+2)(s^2+5) + 4(s+2)}{s^4 + 13s^2 + 36} = \frac{(s+2)(s^2+9)}{(s^2+9)(s^2+4)} = \frac{s+2}{s^2+4}.$$

Therefore

$$\boxed{y(t) =} \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} + \frac{2}{s^2+4} \right\} = \boxed{\cos(2t) + \sin(2t).}$$

Substituting in $y''(t) = x(t) - 8y(t)$ gives

$$-4\cos(2t) - 4\sin(2t) = x(t) - 8\cos(2t) - 8\sin(2t)$$

or

$$\boxed{x(t) = 4\cos(2t) + 4\sin(2t)}$$

<u>Substitution Method</u>: To solve $\begin{cases} x'' = -5x + 4y \\ y'' = x - 8y, \end{cases}$

we solve for x in the second equation, obtaining $x = y'' + 8y$, and substitute this expression in the first equation:

$$(y'' + 8y)'' = -5(y'' + 8y) + 4y.$$

Simplifying and rearranging produces

$$y^{(4)} + 8y'' = -5y'' - 40y + 4y$$

or $\quad y^{(4)} + 13y'' + 36y = 0$. Then $y(t) = e^{rt}$ leads to

$r^4 + 13r^2 + 36 = 0$. Factoring gives $(r^2+9)(r^2+4) = 0$ so

$r = \pm 3i$ or $r = \pm 2i$. Therefore $y(t) = c_1\cos(3t) + c_2\sin(3t) + c_3\cos(2t) + c_4\sin(2t)$

is the general solution of $y^{(4)} + 13y'' + 36y = 0$. observe that the given initial conditions and the relation $x = y'' + 8y$ imply

$$y(0) = 1, \quad y'(0) = 2,$$

$$y''(0) = x(0) - 8y(0) = 4 - 8(1) = -4$$

$$y'''(0) = x'(0) - 8y'(0) = 8 - 8(2) = -8.$$

Applying these initial conditions to the general solution

$$y(t) = c_1\cos(3t) + c_2\sin(3t) + c_3\cos(2t) + c_4\sin(2t)$$

and its derivatives

$$y'(t) = -3c_1\sin(3t) + 3c_2\cos(3t) - 2c_3\sin(2t) + 2c_4\cos(2t)$$

$$y''(t) = -9c_1\cos(3t) - 9c_2\sin(3t) - 4c_3\cos(2t) - 4c_4\sin(2t)$$

$$y'''(t) = 27c_1\sin(3t) - 27c_2\cos(3t) + 8c_3\sin(2t) - 8c_4\cos(2t),$$

we find

$$\begin{cases} 1 = c_1 + c_3 \\ 2 = 3c_2 + 2c_4 \\ -4 = -9c_1 - 4c_3 \\ -8 = -27c_2 - 8c_4 \end{cases}$$

and hence $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, $c_4 = 1$. That is

$$\boxed{y(t) = \cos(2t) + \sin(2t)}.$$

As in the Laplace transform method we find $x(t)$ by substituting in $y'' = x - 8y$ to obtain

$$\boxed{x(t) = 4\cos(2t) + 4\sin(2t)}.$$

Matrix Method: We express the system in vector-matrix form as

$$\vec{x}\,'' = A\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ \vec{x}\,'(0) = \begin{bmatrix} 8 \\ 2 \end{bmatrix},$$

where $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} -5 & 4 \\ 1 & -8 \end{bmatrix}$. We assume a solution of the form $\vec{x} = \vec{k}e^{rt}$ where r and $\vec{k}$ are constants. Then $\vec{x}\,' = r\vec{k}e^{rt}$ and $\vec{x}\,'' = r^2\vec{k}e^{rt}$ so substituting in $\vec{x}\,'' = A\vec{x}$ and canceling e^{rt} from both sides yields $r^2\vec{k} = A\vec{k}$. This is the eigenvalue equation for A with eigenvalue $\lambda = r^2$. The eigenvalues of A satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 4 \\ 1 & -8-\lambda \end{vmatrix} = (5+\lambda)(8+\lambda) - 4 = \lambda^2 + 13\lambda + 36$$

which factors as

$$0 = (\lambda+9)(\lambda+4)$$

so $\lambda = -9$ or $\lambda = -4$. The corresponding eigenvectors satisfy $(A - \lambda I)\vec{k} = \vec{0}$. For $\lambda = -9$ this becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \iff \begin{cases} 0 = 4k_1 + 4k_2 \\ 0 = k_1 + k_2 \end{cases} \Rightarrow k_2 = -k_1$$

Therefore $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -9$. Then $r^2 = \lambda = -9$ so $r = \pm 3i$ and hence

$$(*) \quad \vec{x}(t) = \vec{k}e^{rt} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\pm 3it}$$

are solutions to $\vec{x}\,'' = A\vec{x}$.

For $\lambda = -4$, the corresponding eigenvectors satisfy

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}-1 & 4\\ 1 & -4\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} \iff \begin{cases} 0 = -k_1 + 4k_2 \\ 0 = k_1 - 4k_2 \end{cases} \Rightarrow k_1 = 4k_2 .$$

Thus $\vec{k}^{(2)} = \begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}4k_2\\k_2\end{bmatrix} = k_2\begin{bmatrix}4\\1\end{bmatrix}$ is an eigenvector corresponding to $\lambda = -4$. Then $r^2 = \lambda = -4$ so $r = \pm 2i$ and hence

$$(**) \qquad \vec{x}(t) = \vec{k}e^{rt} = \begin{bmatrix}4\\1\end{bmatrix}e^{\pm 2it}$$

are solutions to $\vec{x}'' = A\vec{x}$. Taking real and imaginary parts in equations $(*)$ and $(**)$ we see that

$$\vec{x}^{(1)}(t) = \begin{bmatrix}1\\-1\end{bmatrix}\cos(3t)$$

$$\vec{x}^{(2)}(t) = \begin{bmatrix}1\\-1\end{bmatrix}\sin(3t)$$

$$\vec{x}^{(3)}(t) = \begin{bmatrix}4\\1\end{bmatrix}\cos(2t)$$

$$\vec{x}^{(4)}(t) = \begin{bmatrix}4\\1\end{bmatrix}\sin(2t)$$

are vector solutions to $\vec{x}'' = A\vec{x}$ with real components. It is not hard to see that

$$\vec{x}(t) = \begin{bmatrix}1\\-1\end{bmatrix}\left(c_1\cos(3t) + c_2\sin(3t)\right) + \begin{bmatrix}4\\1\end{bmatrix}\left(c_3\cos(2t) + c_4\sin(2t)\right)$$

is the general solution to $\vec{x}'' = A\vec{x}$ where c_1, c_2, c_3, c_4 are arbitrary constants. Applying the initial conditions gives

$$\begin{bmatrix}4\\1\end{bmatrix} = \vec{x}(0) = \begin{bmatrix}1\\-1\end{bmatrix}c_1 + \begin{bmatrix}4\\1\end{bmatrix}c_3 \quad \text{and} \quad \begin{bmatrix}8\\2\end{bmatrix} = \vec{x}'(0) = \begin{bmatrix}1\\-1\end{bmatrix}(3c_2) + \begin{bmatrix}4\\1\end{bmatrix}(2c_4)$$

so $c_1 = 0 = c_2$ and $c_3 = 1 = c_4$. That is,

$$\boxed{\begin{bmatrix}x(t)\\y(t)\end{bmatrix}} = \vec{x}(t) = \begin{bmatrix}4\\1\end{bmatrix}\left(\cos(2t) + \sin(2t)\right) = \boxed{\begin{bmatrix}4\cos(2t) + 4\sin(2t)\\ \cos(2t) + \sin(2t)\end{bmatrix}} .$$

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}$, $n = 0,1,2,3\ldots.$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}

Math 204 Final Exam "Master List" Scorecard, 2012 Spring Semester

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Number taking final: 297

Median: 138

Mean: 136.2

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