Mathematics 204

Fall 2013

Exam I

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 7 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

problem	0	1	2	3	4	5	6	7	**Sum**
points earned									
maximum points	2	14	14	14	14	14	14	14	100

1. [14] Determine all values of r for which the differential equation $9t^2y'' - 3ty' + 4y = 0$ has solutions of the form $y = t^r$ on the interval $t > 0$.

If $y = t^r$ then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. We want $y = t^r$ to solve the DE, so we must have

$$9t^2 r(r-1)t^{r-2} - 3trt^{r-1} + 4t^r = 0 \qquad \text{for } t > 0.$$

Simplifying,

$$9r(r-1)t^r - 3rt^r + 4t^r = 0$$

$$(9r^2 - 9r - 3r + 4)t^r = 0$$

$$9r^2 - 12r + 4 = 0$$

$$(3r-2)(3r-2) = 0$$

$$\therefore \boxed{r = \frac{2}{3}}$$

That is, $y = t^{2/3}$ solves the DE on $t > 0$.

2. [14] Find the explicit solution of the differential equation $ty' = y^2 + 1$.

This equation is first-order and separable. Rewrite it as

$$t\frac{dy}{dt} = y^2+1$$

and separate variables:

$$\frac{dy}{y^2+1} = \frac{dt}{t}.$$

Integrating both sides gives

$$\operatorname{Arctan}(y) = \int \frac{dy}{y^2+1} = \int \frac{dt}{t} = \ln|t| + c.$$

To solve for y, we take the tangent of both sides:

$$y = \tan(\operatorname{Arctan} y) = \tan(\ln|t| + c).$$

Therefore $\boxed{y(t) = \tan(\ln|t| + c)}$ is the explicit general solution of the DE.

3. [14] Solve the initial value problem $ty' = y + 2t^2$, $y(2) = 10$.

This DE is first-order and linear. Rewriting it yields

$$ty' - y = 2t^2,$$

and placing it in standard form, we have

$$(*) \qquad y' + \frac{-1}{t} y = 2t.$$

An integrating factor is

$$e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln|t| + \cancel{c}^{0}} = e^{\ln|t|^{-1}} = \frac{1}{|t|}.$$

Since we want to solve the DE in a neighborhood of $t=2$, we will assume $t>0$ so $\frac{1}{|t|} = \frac{1}{t}$. Multiplying through equation $(*)$ by the integrating factor produces

$$\frac{1}{t}y' - \frac{1}{t^2}y = 2.$$

Observe that the left member of this equation is exact: $\left(\frac{1}{t}y\right)' = \frac{1}{t}y' + \frac{-1}{t^2}y$.
Therefore

$$\left(\frac{1}{t}y\right)' = 2,$$

so integrating both sides gives

$$\frac{1}{t}y = \int \left(\frac{1}{t}y\right)' dt = \int 2\,dt = 2t + c.$$

Solving for y yields

$$y = 2t^2 + ct.$$

Applying the initial condition, we have $10 = y(2) = 2(2)^2 + c(2)$ so $c = 1$.

Consequently $\boxed{y(t) = 2t^2 + t}$ solves the IVP.

4. A tank initially contains 100 gallons of water in which 10 pounds of salt is dissolved. A mixture containing 2 pounds of salt per gallon of water is pumped into the tank. At time $t > 0$, this mixture is pumped in at a rate of

$$3 - \frac{t}{10}$$

gallons per minute. The well-mixed solution is pumped out at a rate of three gallons per minute.

(a) [11] Write, **BUT DO NOT SOLVE**, an initial value problem for the amount $A(t)$ of salt in the tank at time t.

Net rate = Rate in − Rate out.

$$\frac{dA}{dt} = \left(\frac{2 \text{ lbs.}}{\cancel{\text{gal.}}}\right)\left(3 - \frac{t}{10}\right)\left(\frac{\cancel{\text{gal.}}}{\text{min}}\right) - \left(\frac{3\,\cancel{\text{gal.}}}{\text{min.}}\right)\left(\frac{A(t) \text{ lbs.}}{V(t)\,\cancel{\text{gal.}}}\right)$$

The volume $V(t)$ of solution in the tank at time t obeys the relation

$$\frac{dV}{dt} = \overbrace{\left(3 - \frac{t}{10}\right)}^{\text{rate in}} - \overbrace{3}^{\text{rate out}} = -\frac{t}{10} \text{ gal./min.}$$

Hence $V(t) = \int -\frac{t}{10}\,dt = -\frac{t^2}{20} + V(0) = -\frac{t^2}{20} + 100$. Therefore

$$\boxed{\frac{dA}{dt} = 2\left(3 - \frac{t}{10}\right) - \frac{3A}{100 - \frac{t^2}{20}}, \quad A(0) = 10} \quad (A \text{ in pounds, } t \text{ in minutes})$$

is an IVP that models the amount $A(t)$ of salt in the tank at time t.

(b) [3] Find the time when the tank becomes empty.

$$0 = V(t) = 100 - \frac{t^2}{20} \Rightarrow t^2 = 20(100) \Rightarrow \boxed{t = \sqrt{2000} \text{ minutes}}$$

5. [14] Consider the differential equation $y' = y(2-y)(5-y)^2$.

(a) Find the equilibrium (or critical) points.

(b) Sketch the phase line (or phase portrait). Be sure to **SHOW YOUR WORK**.

(c) Classify each equilibrium point as asymptotically stable, unstable, or semi-stable.

(d) If $y(0)=3$, find the limit as t goes to infinity of the solution $y(t)$.

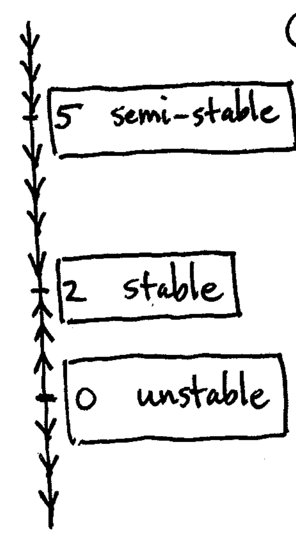
(a) $0 = y' = y(2-y)(5-y)^2$ so $\boxed{y=0,\ y=2, \text{ and } y=5}$ are the equilibrium points of the DE.

(b)

intervals of y-values	sign of $y' = y(2-y)(5-y)^2$
$-\infty < y < 0$	$(-)(+)(+) = -$
$0 < y < 2$	$(+)(+)(+) = +$
$2 < y < 5$	$(+)(-)(+) = -$
$5 < y < \infty$	$(+)(-)(+) = -$

Phase line

(c)



(d) Since $y(0) = 3$ is in the interval $2 < y < 5$, the solution $y(t)$ will decrease to the stable equilibrium point $y = 2$. That is,

$$\lim_{t \to \infty} y(t) = \boxed{2}.$$

6. [14] Find the general solution of each differential equation.

(a) $y'' + 2y' - y = 0$

$y = e^{rt}$ leads to $r^2 + 2r - 1 = 0$ so $r = \dfrac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)} = \dfrac{-2 \pm \sqrt{8}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$. Therefore, the general solution is

$$\boxed{y(t) = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}}$$

where c_1 and c_2 are arbitrary constants.

(b) $y'' + 2y' + 4y = 0$

$y = e^{rt}$ leads to $r^2 + 2r + 4 = 0$ so $r = \dfrac{-2 \pm \sqrt{4 - 4(1)(4)}}{2(1)} = \dfrac{-2 \pm \sqrt{-12}}{2}$

$= \dfrac{-2 \pm 2\sqrt{3}\,i}{2} = -1 \pm \sqrt{3}\,i$. Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t)}$$

where c_1 and c_2 are arbitrary constants.

7. [14] Given that $y_1 = e^t$ is a solution of the differential equation

$$ty'' + (1-2t)y' + (t-1)y = 0$$

on the interval $t > 0$, use reduction of order to find a second linearly independent solution y_2.

Assume $y_2(t) = u(t)y_1(t) = u(t)e^t$ where u is a nonconstant function. Then $y_2' = u'(t)e^t + u(t)e^t$ and $y_2'' = u''(t)e^t + 2u'(t)e^t + u(t)e^t$. We want to choose u such that y_2 solves the DE:

$$ty_2'' + (1-2t)y_2' + (t-1)y_2 = 0.$$

Substituting the expressions for y_2 and its derivatives gives

$$t[u''e^t + 2u'e^t + ue^t] + (1-2t)[u'e^t + ue^t] + (t-1)ue^t = 0.$$

Dividing through by e^t and then simplifying yields

$$t[u'' + 2u' + u] + (1-2t)[u' + u] + (t-1)u = 0$$

$$tu'' + (\underbrace{2t + 1 - 2t}_{1})u' + (\underbrace{t + 1 - 2t + t - 1}_{0})u = 0$$

$$tu'' + u' = 0.$$

Let $v = u'$. Then $v' = u''$ so the above DE becomes $tv' + v = 0$. This equation is first-order and separable. Rewriting gives

$$t\frac{dv}{dt} = -v \qquad \text{so} \qquad \frac{dv}{v} = -\frac{dt}{t}.$$

Integrating produces $\ln|v| = \int \frac{dv}{v} = \int -\frac{dt}{t} = -\ln|t| + c_1$. Solving for v (by exponentiating both sides) yields

$$v = \pm e^{-\ln|t| + c_1} = k_1 e^{\ln|t|^{-1}} = \frac{k_1}{t} \quad (\text{where } k_1 = \pm e^{c_1}).$$

But $v = u'$ so $\quad u = \int \frac{k_1}{t}\,dt = k_1\ln(t) + k_2.$

Thus $y_2(t) = u(t)y_1(t) = (k_1\ln(t) + k_2)e^t = k_1e^t\ln(t) + k_2e^t$. Taking $k_1 = 1$, $k_2 = 0$ yields a second linearly independent solution $\boxed{y_2(t) = e^t\ln(t)}$ on $t > 0$.

(Clearly $y_2(t)/y_1(t) = \ln(t) \neq$ constant, so y_2 and y_1 are linearly indep.)

2013 Fall Semester, Math 204 Hour Exam I
Instructor Grow, Section D

Score	Tally	Grade	Score	Tally	Grade	Score
100			59			19
99			58			18
98			57			17
97			56			16
96	II		55			15
95	II		54			14
94	II	10 As	53		4 Fs	13
93	I		52			12
92	II		51			11
91			50			10
90	I		49	I		9
89	I		48			8
88			47	I		7
87	I		46			6
86	I		45			5
85		6 Bs	44			4
84	I		43			3
83			42			2
82	I		41	I		1
81			40	I		0
80	I		39			
79	III		38			
78	II		37			
77	I		36			
76	I		35			
75	I	11 Cs	34			
74	I		33			
73			32			
72	I		31			
71			30			
70	I		29			
69			28			
68			27			
67			26			
66			25			
65	I	4 Ds	24			
64			23			
63	I		22			
62			21			
61	I		20			
60	I					

Number taking exam: 35
Median: 79
Mean: 77.4
Standard Deviation: 15.6

Number receiving A's: 10 28.6%
Number receiving B's: 6 17.1
Number receiving C's: 11 31.4
Number receiving D's: 4 11.4
Number receiving F's: 4 11.4