**Mathematics 204**

**Spring 2013**

**Exam I**

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 4 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

| **problem** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | **Sum** |
|---|---|---|---|---|---|---|---|---|---|
| **points earned** | | | | | | | | | |
| **maximum points** | 2 | 14 | 12 | 14 | 11 | 14 | 15 | 18 | 100 |

1. Find the general solution of each differential equation.

[7] (a) $y''-5\pi^2 y=0$ $\quad y=e^{rt}$ leads to $r^2-5\pi^2=0$ so $r=\pm\pi\sqrt{5}$.

The general solution is $\boxed{y(t)=c_1e^{\pi\sqrt{5}t}+c_2e^{-\pi\sqrt{5}t}}$ where $c_1$ and $c_2$ are arbitrary constants.

[7] (b) $y''+y'+y=0$ $\quad y=e^{rt}$ leads to $r^2+r+1=0$ so $r=\dfrac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.

The general solution is $\boxed{y(t)=e^{-t/2}\left(c_1\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}$ where $c_1$ and $c_2$ are arbitrary constants.

2.[12] Solve the boundary value problem $y''-4y'+4y=0,\quad y(0)=5,\ y\left(\dfrac{\ln(5)}{2}\right)=0.$

$y=e^{rt}$ in the DE leads to $r^2-4r+4=0 \Rightarrow (r-2)^2=0$ so $r=2$ with multiplicity two. The general solution of the DE is $y(t)=c_1e^{2t}+c_2te^{2t}$ where $c_1$ and $c_2$ are arbitrary constants. We now apply the two boundary conditions.

$$5=y(0)=c_1e^0+c_2(0)e^0=c_1$$

$$0=y\left(\frac{\ln 5}{2}\right)=c_1e^{\ln(5)}+c_2\left(\frac{\ln 5}{2}\right)e^{\ln 5}=5\left(c_1+\frac{c_2\ln 5}{2}\right)\Rightarrow c_2=\frac{-2c_1}{\ln(5)}$$

$$=\frac{-10}{\ln(5)}$$

Therefore $\boxed{y(t)=5e^{2t}-\frac{10te^{2t}}{\ln(5)}}$ solves the BVP.

3. Consider the differential equation $y' = 120y - 0.1y^2$.

[5] (a) Determine the critical (or equilibrium) points of the differential equation.

[5] (b) Draw the phase line of the differential equation. Be sure to **SHOW YOUR WORK.**

[4] (c) Classify each critical point as asymptotically stable, unstable, or semistable.

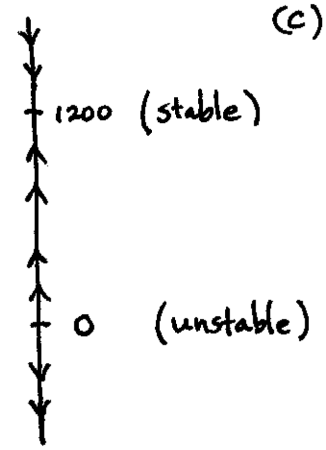
(a) Critical points of the autonomous DE are constant solutions $y(t) = c$. Then $y'(t) = 0$ for all $t$ so substituting in the DE gives $0 = 120c - 0.1c^2 = (1200 - c)(0.1)c$ so $\boxed{y = 0 \text{ and } y = 1200}$ are the critical points.

(b)

| Interval | Sign of $y' = (1200-y)(0.1)y$ |
|---|---|
| $1200 < y < \infty$ | $-$ |
| $0 < y < 1200$ | $+$ |
| $-\infty < y < 0$ | $-$ |

Phase Line:

(c)




4.[11] Classify the following differential equations by filling in the corresponding table entries.

| | order? | linear? | homogeneous? |
|---|---|---|---|
| $4y'' - 3\sin(t)\,y' + 8t^2 = 0$ | 2 | Yes | No |
| $yy' + y^{(4)}\sec(t) + 12t = 0$ | 4 | No | (not applicable) |
| $8 + \dfrac{t}{t+1}y' + y\sqrt{t} = 0$ | 1 | Yes | No |
| $t\dfrac{d^2}{dt^2}\left(\dfrac{dy}{dt} + ty\right) = 4y$ | 3 | Yes | Yes |

Equivalent to $ty''' + t(ty)'' = 4y$

5.[14] Find an explicit general solution of the differential equation $y' - ty^2 = t$. ← ($1^{st}$ order, nonlinear)

We separate variables: $\frac{dy}{dt} = ty^2 + t = t(y^2+1)$ so $\frac{dy}{y^2+1} = t\,dt$.

Integrating both sides yields $\text{Arctan}(y) = \frac{t^2}{2} + c$. To solve for $y$ we take the tangent of both sides:

$$\boxed{y(t) = \tan\left(\frac{t^2}{2} + c\right)}$$

where $c$ is an arbitrary constant.

6. A trout population in a remote stream at time $t$ has a birth rate that is 120 times the population at that instant and a death rate that is equal to one tenth of the square of the population at that instant.

[11] (a) Write, BUT DO NOT SOLVE, an initial value problem that models the trout population at any time $t > 0$ if the stream was initially stocked with 4000 trout.

Let $P(t)$ denote the trout population in the stream at time $t$. We use the flow principle, Net Rate = Rate In − Rate Out. +1

Then

$$\boxed{\frac{dP}{dt} = 120\,P(t) - 0.1\,(P(t))^2 , \quad P(0) = 4000} \quad +10$$

(3) (3) (3) (1)

is an IVP that models the trout population.

[4] (b) Without solving the initial value problem, predict the trout population in the stream after a long time and justify your answer.

+1

After a long time, the trout population will be approximately 1200. To see this, note that the IVP above has the same DE as in problem 3. +1 From problem 3 we see that if the initial value is $(P(0)=)\ y(0) = 4000$ then the solution $(P(t)=)\ y(t)$ will decrease to the stable equilibrium +2 point $(P=)\ y = 1200$.

7. Consider the initial value problem $ty' - y = t + \frac{1}{t}$, $y(1) = 0$. (1st order, linear, nonhomogeneous)

[14] (a) Solve the initial value problem.
[2] (b) Determine the largest interval in which the solution is defined.
[2] (c) Determine how the solution behaves as $t \to \infty$.

(a) We first normalize the DE: $y' - \frac{1}{t}y = 1 + \frac{1}{t^2}$.

An integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int -t^{-1}dt} = e^{-\ln|t| + \cancel{c}^{\,0}} = e^{\ln|t|^{-1}} = |t|^{-1}.$$

We will solve the IVP henceforth on the interval $0 < t < \infty$ (since the initial condition involves $t_0 = 1 > 0$), so we may drop absolute values: $\mu(t) = t^{-1}$.
Multiplying the normalized DE by the integrating factor yields

$$t^{-1}(y' - t^{-1}y) = t^{-1}(1 + t^{-2})$$

$$t^{-1}y' - t^{-2}y = t^{-1} + t^{-3}.$$

Note that the left member of the last equation is exact: $\frac{d}{dt}(t^{-1}y) = t^{-1}y' - t^{-2}y$.
Substituting and then integrating both sides produces

$$\frac{d}{dt}(t^{-1}y) = t^{-1} + t^{-3}$$

$$t^{-1}y = \int (t^{-1} + t^{-3})dt = \ln(t) - \frac{1}{2t^2} + c.$$

Multiplying both sides by $t$ gives $y = t\ln(t) - \frac{1}{2t} + ct$. Applying the initial condition, we find $0 = y(1) = 1\ln(1) - \frac{1}{2} + c$ so $c = \frac{1}{2}$. Thus

$$\boxed{y(t) = t\ln(t) - \frac{1}{2t} + \frac{t}{2}}$$

solves the IVP.

(b) The largest interval in which the solution is defined is $\boxed{0 < t < \infty.}$

(c) As $t \to \infty$, $-\frac{1}{2t} \to 0$ and $t\ln|t| + \frac{t}{2} \to +\infty$. Thus

$$\boxed{y(t) \to \infty \quad \text{as} \quad t \to \infty}.$$

**2013 Spring Semester, Math 204 Hour Exam I**
**Instructor Grow, Section E**

| | | | | | | |
|---|---|---|---|---|---|---|
| 100 | I | | 59 | I | | 19 |
| 99 | II | | 58 | I | | 18 |
| 98 | | | 57 | II | | 17 |
| 97 | | | 56 | I | | 16 |
| 96 | | 9 As | 55 | I | 12 Fs | 15 |
| 95 | | | 54 | | | 14 |
| 94 | I | | 53 | I | | 13 |
| 93 | I | | 52 | I | | 12 |
| 92 | II | | 51 | I | | 11 |
| 91 | | | 50 | | | 10 |
| 90 | II | | 49 | I | | 9 |
| 89 | | | 48 | | | 8 |
| 88 | I | | 47 | | | 7 |
| 87 | II | | 46 | | | 6 |
| 86 | I | | 45 | | | 5 |
| 85 | | 10 Bs | 44 | | | 4 |
| 84 | I | | 43 | | | 3 |
| 83 | II | | 42 | I | | 2 |
| 82 | I | | 41 | | | 1 |
| 81 | | | 40 | | | 0 |
| 80 | II | | 39 | | | |
| 79 | I | | 38 | | | |
| 78 | I | | 37 | | | |
| 77 | II | | 36 | | | |
| 76 | I | | 35 | | | |
| 75 | II | 11 Cs | 34 | I | | |
| 74 | | | 33 | | | |
| 73 | I | | 32 | | | |
| 72 | I | | 31 | | | |
| 71 | I | | 30 | | | |
| 70 | I | | 29 | | | |
| 69 | | | 28 | | | |
| 68 | | | 27 | | | |
| 67 | | | 26 | | | |
| 66 | II | | 25 | | | |
| 65 | | 4 Ds | 24 | | | |
| 64 | I | | 23 | | | |
| 63 | | | 22 | | | |
| 62 | I | | 21 | | | |
| 61 | | | 20 | | | |
| 60 | | | | | | |

Number taking exam: 46
Median: 76.5
Mean: 73.8
Standard Deviation: 16.2

Number receiving A's: 9 19.6%
Number receiving B's: 10 21.7
Number receiving C's: 11 23.9
Number receiving D's: 4 8.7
Number receiving F's: 12 26.1