Mathematics 204

Summer 2013

Exam I

[1] Your Printed Name: Solution Key

[1] Your Instructor's Name: Streipert

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam I consists of this cover page and 6 pages of problems containing 7 numbered problems.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

	0	1	2	3	4	5	6	7	**Sum**
points earned									
maximum points	2	12	14	17	17	12	10	16	100

1.[12] Determine the order of each differential equation and decide also whether it is linear and/or homogeneous. For each nonlinear equation, circle a term that makes the equation nonlinear.

	Differential Equation	Order	Linear	Homogeneous
1)	$(1-x)-4xy'+5y^{(5)}=0$	5th	linear	Nonhomogeneous
2)	$ty'+yy''=\sin(t)$	2nd	Nonlinear	N.A.
3)	$\frac{d^2y}{dx^2}+\tan(x)y=\sqrt{\frac{dy}{dx}}$	2nd	Nonlinear	N.A.
4)	$\pi\frac{d^3y}{dx^3}+y=\csc(x^2)\left(\frac{dy}{dx}\right)$	3rd	Linear	Nhomogeneous

2.[14] Find the equilibrium solutions and sketch the phase portrait of the differential equation $y'=2y^2(9-y^2)$. Classify each equilibrium solution as asymptotically stable, unstable, or semi-stable.

a) Find equilibrium solutions: $y'=0$ for $y=0,+3,-3$

b)&c)

Phase dine

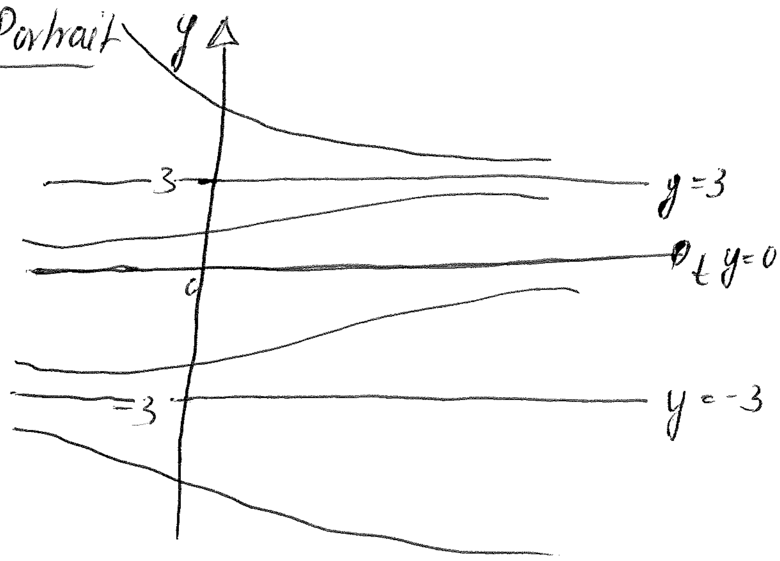
y	y'	y
$(-\infty,-3)$	<0	decr.
$(-3,0)$	>0	incr.
$(0,3)$	>0	incr.
$(3,\infty)$	$<\infty$	decr.

$y=3$ is asymptotically stable

$y=0$ is — " — semistable

$y=-3$ is — " — unstable

Phase Portrait



3.[17] Solve the differential equation $ty' + 2y = \frac{\sin t}{t}$ with $y(\pi)=0$ for $t > 0$.

$t^2 y' + 2ty = \sin t$ $\quad$ 1^{st} order, linear $\Rightarrow$ use integrating Factor Method.

Step 1 $\quad y' + \frac{2t}{t^2} y = \frac{\sin t}{t^2}$

$$y' + \underbrace{\frac{2}{t}}_{p(t)} y = \frac{\sin t}{t^2}$$

Step 2 $\quad \mu(t) = \exp\{\int p(t)\,dt\} = \exp\{\int \frac{2}{t}\,dt\} = e^{2\int \frac{1}{t}dt} = e^{2\ln t} = t^2$

Step 3 $\quad t^2\left(y' + \frac{2}{t}y\right) = t^2 \frac{\sin t}{t^2}$

$$(t^2 y)' = \sin t$$

Step 4 $\quad \int (t^2 y)'\,dt = \int \sin t\,dt$

$$t^2 y = -\cos t + C$$

$$y = -\frac{\cos t}{t^2} + \frac{C}{t^2}$$

Step 5 $\quad 0 = y(\pi) = -\frac{\overbrace{\cos(\pi)}^{=-1}}{\pi^2} + \frac{C}{\pi^2} = \frac{-(-1)+C}{\pi^2} = \frac{1+C}{\pi^2}$

$$\Rightarrow \quad 0 = 1 + C \quad \Rightarrow \quad C = -1$$

Step 6 $\quad y(t) = -\frac{\cos t}{t^2} - \frac{1}{t^2} = -t^{-2}(\cos t + 1)$

4.[17] Find the explicit solution of the initial value problem $(1+t^2)y' - 2ty^2 = 0, \quad y(0) = -2.$

$$y' = \frac{2ty^2}{1+t^2}$$ 1st order, nonlinear

Step 1

$$\frac{dy}{dt} = y' = \underbrace{\frac{2t}{1+t^2}}_{h(t)} \cdot \underbrace{y^2}_{g(y)}$$

Step 2

$$\frac{1}{y^2}\,dy = \frac{2t}{1+t^2}\,dt$$

Step 3

$$\int \frac{1}{y^2}\,dy = \int \frac{2t}{1+t^2}\,dt$$

$$-y^{-1} = \ln(1+t^2) + C$$ (implicit solution)

Step 4

$$y(t) = \frac{-1}{\ln(1+t^2) + C}$$

$$-2 = y(0) = \frac{-1}{\underbrace{\ln(1+0^2)}_{\ln(1)=0} + C} = -\frac{1}{C} \Rightarrow \underline{C = 1/2}$$

Step 5

$$y(t) = \frac{-1}{\ln(1+t^2) + 1/2}$$

5.[12] a) One theory of epidemic spread postulates that the amount of infected individuals in a certain area increases at a rate proportional to the current infected individuals at time t, in the absence of any other factors. Scientists have now found a medication that heals 15 infected individuals a day. Set up (BUT DO NOT SOLVE) an initial value problem for the amount of infected individuals I(t) for time t≥0, assuming that the proportionality constant is 0.5 per week and 150 individuals were initially infected.

Step1: Absence of other factors: $r > 0$

$$\frac{dI(t)}{dt} = r \cdot I(t)$$

Step2 given information

$$\frac{dI(t)}{dt} = 0.5 \cdot I(t) - \overbrace{15 \cdot 7}^{\text{per week}}$$

$$I(0) = 150$$

b) Determine whether your differential equation you set up in a) is linear or nonlinear and name the method you would use to solve it. (DO NOT SOLVE IT).

linear, (1^{st} order, nonhomogeneous)

use integrating factor method $I' - 0.5I = -15 \cdot 7$

(or separation of variables method)

$$\frac{dI}{dt} = g(I) \cdot h(t)$$

$$h(t) = 1 \quad g(I) = \left(\tfrac{1}{2}I - 15 \cdot 7\right)$$

6.[10] Find the most general solution to the differential equation $y'' + 2y' - 3y = 0$

2nd order, linear, homogeneous, constant coefficients

$\Rightarrow y(t) = e^{rt}$

<u>Plug in:</u>

$\Rightarrow r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$

$e^{rt}(r^2 + 2r - 3) = 0$

$\underbrace{r^2 + 2r - 3}_{(r-1)(r+3)} = 0$

or alternative: $r_{1,2} = \frac{-2 \pm \sqrt{4 - 4\cdot(-3)}}{2} = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2}$

$\Rightarrow r_1 = 1$

$r_2 = -3$

$r_1 = \frac{-2+4}{2} = 1$

$r_2 = \frac{-2-4}{2} = -3$

$y_1(t) = e^{t}, \quad y_2(t) = e^{-3t}$

$\boxed{\text{most general solution is then: } y(t) = c_1 y_1 + c_2 y_2 = c_1 \cdot e^{t} + c_2 e^{-3t}}$, since y_1 & y_2 are linearly independent: <u>Check</u>

$W(y_1, y_2) = \begin{vmatrix} e^{t} & e^{-3t} \\ e^{t} & -3e^{-3t} \end{vmatrix} = -3e^{-3t} \cdot e^{t} - e^{-3t} \cdot e^{t} = -3e^{-2t} - e^{-2t} = -4e^{-2t}$ $\neq 0$

<u>Note:</u> $W(y_1, y_2) = -4e^{-2t} \neq 0$ for all t.

7.[16] Determine the largest interval in which a unique solution of the given initial value problem is certain to exist. $(\ln t)y' + (t-3)y = t$ with y(2)=1.

$$y' + \underbrace{\frac{(t-3)}{\ln t}}_{p(t)} y = \underbrace{\frac{t}{\ln t}}_{q(t)} \qquad y(2)=1$$

$p(t)$ is continuous where $\ln t$ exists $\Rightarrow t>0$
and where denominator is not equal zero
so $\ln(t) \neq 0$ for $t \neq 1$

$q(t)$ is continuous also where $\ln t$ exists $\Rightarrow t>0$
and $\ln t \neq 0 \rightarrow t \neq 1$

Therefore we have two possible intervals
$(0,1)$, $(1,\infty)$ Since $t_0=2$ has to be in
the interval, the largest interval is $\underline{(1,\infty)}$.

2013 Summer Semester, Math 204 Hour Exam _1___
Instructor __Sabrina Streipert___, Section ___A__

100	1	59		19	
99		58		18	
98	2	57	2	17	
97	2	56		16	
96	1	55		15	
95	3	54		14	
94		53	1	13	
93	2	52		12	
92	1	51		11	
91	1	50		10	
90		49		9	
89		48		8	
88	3	47		7	
87	1	46		6	
86	3	45		5	
85	4	44		4	
84	3	43		3	
83	3	42		2	
82	3	41		1	
81	1	40	1	0	
80	2	39			
79	1	38			
78	1	37			
77	2	36			
76		35			
75	1	34			
74		33			
73	2	32			
72	1	31			
71	3	30			
70	1	29			
69		28			
68		27			
67		26	1		
66	1	25			
65		24			
64		23			
63		22			
62		21			
61	1	20			
60					

Number taking exam:_55__
Median: __83.0___
Mean: __79.0____
Standard Deviation: 17.8___

Number receiving A's:____13___
Number receiving B's:____23__
Number receiving C's:_____12__
Number receiving D's:____2___
Number receiving F's:____5___