Mathematics 204

Spring 2013

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page and 5 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	20	14	20	26	20	100

1.[20] Solve the following differential equations.

(a) $y^{(6)} - 16y'' = 0$ $\quad y = e^{rt}$ in this DE leads to $r^6 - 16r^2 = 0 \Rightarrow r^2(r^4-16)=0$

$\Rightarrow r^2(r^2-4)(r^2+4) = 0 \quad \Rightarrow r^2(r-2)(r+2)(r^2+4) = 0$. Therefore the roots to the characteristic equation are $r = 0$ (multiplicity 2), $r = 2$, $r = -2$, and $r = \pm 2i$. Thus

$$\boxed{y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} + c_5\cos(2t) + c_6\sin(2t)}$$

($c_1, c_2, c_3, c_4, c_5, c_6$ arbitrary constants)

is the general solution of the DE.

(b) $t^2y'' + 9y = 0$ $\quad y = t^m$ in this DE leads to $m(m-1) + 9 = 0 \Rightarrow m^2 - m + 9 = 0$.

Thus $m = \dfrac{1 \pm \sqrt{1-36}}{2} = \dfrac{1}{2} \pm \dfrac{i\sqrt{35}}{2}$. Therefore the general solution of the DE for $t > 0$ is

$$\boxed{y = c_1 t^{1/2}\cos\left(\frac{\sqrt{35}}{2}\ln(t)\right) + c_2 t^{1/2}\sin\left(\frac{\sqrt{35}}{2}\ln(t)\right)}$$

(c_1, c_2 arbitrary constants).

2.[14] What is the largest interval in which the initial value problem

$$(t-\pi)\ln(t)\,y^{(4)} + t\ln(t)\,y''' + t(t-\pi)\,y'' - y' = \sin(t), \quad y(1)=8,\ y'(1)=-2,\ y''(1)=1,\ y'''(1)=-3,$$

is sure to have exactly one solution? Justify your answer.

Normalizing the DE, we find

$$y^{(4)} + \frac{t}{t-\pi}y''' + \frac{t}{\ln(t)}y'' + \frac{-1}{(t-\pi)\ln(t)}y' = \frac{\sin(t)}{(t-\pi)\ln(t)}.$$

The coefficients and the forcing function of the normalized DE are continuous in the following intervals.

coeffs./ $g(t)$	intervals of continuity
$\frac{t}{t-\pi}$	$(-\infty,\pi)$, (π,∞)
$\frac{t}{\ln(t)}$	$(0,1)$, $(1,\infty)$
$\frac{-1}{(t-\pi)\ln(t)}$	$(0,1)$, $(1,\pi)$, (π,∞)
$\frac{\sin(t)}{(t-\pi)\ln(t)}$	$(0,1)$, $(1,\pi)$, (π,∞)

Therefore all the coefficients and the forcing function are continuous in these intervals: $(0,1)$, $(1,\pi)$, (π,∞). However, the initial conditions are given at $t_0=1$ and this point does not belong to any of these three intervals. Consequently, the existence and uniqueness theorem for linear initial value problems does not apply. That is, there is no interval in which the given I.V.P. is sure to have exactly one solution.

3.[20] (Please use 32 ft/sec^2 as the acceleration of gravity in this problem.) A body weighing 4 pounds stretches by 2 inches a spring hanging vertically from a rigid support. The body is given a downward displacement of 2 inches from its static equilibrium position and released with no initial velocity. Assume that no damping forces act on the body.

(a) If an external force at t seconds of $2\cos(3t)$ pounds acts on the body, formulate – BUT DO NOT SOLVE – an initial value problem to describe the motion of the body.

(b) If the given external force is replaced by a force of $4\sin(\omega t)$ pounds with constant angular frequency ω, find the value of ω for which resonance occurs.

(a) $mu'' + \gamma u' + ku = f(t)$

$mg = \text{weight} \Rightarrow m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$ slugs.

$\gamma = 0$ since no damping forces act on the body.

$ku_0 = mg \Rightarrow k = \frac{mg}{u_0} = \frac{4}{1/6} = 24 \ \frac{\text{pounds}}{\text{ft.}}$

$$\boxed{\frac{1}{8}u'' + 24u = 2\cos(3t), \quad u(0) = \frac{1}{6}, \quad u'(0) = 0}$$

(b) $\frac{1}{8}u'' + 24u = 4\sin(wt)$

Resonance occurs when there is no damping and the input frequency ω is equal to the natural frequency ω_0 of the freely oscillating system.

$u = e^{rt}$ in $\frac{1}{8}u'' + 24u = 0$ leads to $\frac{1}{8}r^2 + 24 = 0$ so $r^2 = -192$

$\Rightarrow r = \pm i\sqrt{192} = \pm 8\sqrt{3}\, i$ so $u_c(t) = c_1\cos(8\sqrt{3}\,t) + c_2\sin(8\sqrt{3}\,t)$.

$\therefore \ \omega = \omega_0 = \boxed{8\sqrt{3}}$

is the resonant frequency.

4.[26] Solve the initial value problem $y''+6y'+8y=40\sin(2t),\ y(0)=0,\ y'(0)=0.$

$y=e^{rt}$ in $y''+6y'+8y=0$ leads to $r^2+6r+8=0 \Rightarrow (r+2)(r+4)=0$.
Therefore $r=-2$ and $r=-4$ are the roots of the characteristic equation so
$y_c(t)=c_1e^{-2t}+c_2e^{-4t}$ is the general solution of the associated homogeneous equation.

We use the method of undetermined coefficients to find a particular solution.
A trial particular solution to the nonhomogeneous equation is
$y_p = A\cos(2t)+B\sin(2t)$ where A and B are constants to be determined.
Then $y_p' = -2A\sin(2t)+2B\cos(2t)$ and $y_p''=-4A\cos(2t)-4B\sin(2t)$. We want

$$
\begin{aligned}
40\sin(2t) &= y_p''+6y_p'+8y_p\\
&= [-4A\cos(2t)-4B\sin(2t)]+6[-2A\sin(2t)+2B\cos(2t)]+8[A\cos(2t)+B\sin(2t)]\\
&= (-4A+12B+8A)\cos(2t)+(-4B-12A+8B)\sin(2t)\\
&= (4A+12B)\cos(2t)+(-12A+4B)\sin(2t)
\end{aligned}
$$

Equating like coefficients leads to the system $\begin{cases} 4A+12B=0,\\ -12A+4B=40.\end{cases}$

Multiplying the first equation by 3 and adding the result to the second equation yields $40B=40$ so $B=1$. Substitution in the first equation then gives $A=-3$.
Hence $y_p=-3\cos(2t)+\sin(2t)$. The general solution of DE is

$$y(t)=y_c(t)+y_p(t)=c_1e^{-2t}+c_2e^{-4t}+\sin(2t)-3\cos(2t).$$

Differentiating, we have $y'(t)=-2c_1e^{-2t}-4c_2e^{-4t}+2\cos(2t)+6\sin(2t)$.
Applying the initial conditions gives the system

$$\begin{cases} 0=y(0)=c_1+c_2-3,\\ 0=y'(0)=-2c_1-4c_2+2.\end{cases}$$

Multiplying the first equation by 2 and adding the result to the second equation gives $0=-2c_2-4$ so $c_2=-2$ and thus $c_1=5$. The I.V.P.'s solution is

$$\boxed{y(t)=5e^{-2t}-2e^{-4t}+\sin(2t)-3\cos(2t)}.$$

5.[20] (a) Let $f = f(t)$ be a function defined on the interval $0 \le t < \infty$. State the definition of $\mathcal{L}\{f\}(s)$, the Laplace transform of f at s.

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$ for those values of s for which the improper integral converges. 10 pts.

(−1 if the convergence of the improper integral is omitted)

(b) Recall that the gamma function satisfies $\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx$ for $p > -1$. If $f(t) = t^p$, use the definition of the Laplace transform to help compute $\Gamma(p+1) - \mathcal{L}\{f\}(1)$.

5 pts.

$$\Gamma(p+1) - \mathcal{L}\{f\}(1) = \int_0^\infty e^{-x}x^p\,dx - \int_0^\infty t^p e^{-t}\,dt$$

$$= \int_0^\infty e^{-x}x^p\,dx - \int_0^\infty x^p e^{-x}\,dx$$

because the value of a definite integral does not depend on the (dummy) variable of integration

5 pts.

$$= 0$$

(−1 if have general s instead of 1 in first equation's right member)

2013 Spring Semester, Math 204 Hour Exam II
Instructor Grow, Section E

100
99
98
97
96 I
95 I   5 As
94 I
93
92 I
91 I
90
89 I
88
87
86 I
85   8 Bs
84
83 III
82 II
81
80 I
79 I
78 I
77
76 I
75 II   8 Cs
74
73 I
72
71
70 II
69 IIII
68
67 I
66
65 II   10 Ds
64
63 I
62 I
61 I
60

59 I
58
57 II
56 I
55
54 I   11 Fs
53
52
51
50 I
49
48
47
46
45
44
43
42
41
40 I
39
38
37 I
36
35 I
34
33 I
32 I
31
30
29
28
27
26
25
24
23
22
21
20

19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0

Number taking exam: 42 ✓
Median: 69.5
Mean: 69.1
Standard Deviation: 17.2

Number receiving A's:	5	11.9 %
Number receiving B's:	8	19.0
Number receiving C's:	8	19.0
Number receiving D's:	10	23.8
Number receiving F's:	11	26.2

2013 Spring Semester, Math 204 Hour Exam II, Master List

100
99
98 |||
97 |
96 卌
95 卌 ||||
94 卌 卌
93 |
92 卌 |||
91 卌 |||
90 卌

50 As

89 卌 ||
88 卌 ||
87 卌 卌 |
86 卌 |
85 卌 卌
84 卌 卌
83 卌 卌 |||
82 卌 ||||
81 卌 |
80 卌 |||

87 Bs

79 卌 卌 |||
78 卌 卌
77 卌 卌 |||
76 卌 卌 |
75 卌 |||
74 ||||
73 卌 卌 |
72 卌 卌 |
71 卌 |
70 卌 |

93 Cs

69 卌 |||
68 卌 ||||
67 卌 卌
66 ||||
65 卌 |
64 |||
63 卌 |
62 卌
61 卌 |
60 卌 |

63 Ds

59 卌 |||
58 ||||
57 ||||
56 卌
55 卌 |
54 |||
53 ||
52 |
51 卌 |
50 |||
49 |
48 ||
47 |
46 ||||
45 |||
44 |
43 |
42 |
41
40 ||
39 |
38 |
37 ||||
36
35 ||
34 ||||
33 |
32 |||
31
30
29
28 |
27
26
25 |
24
23
22
21 |
20

79 Fs

19
18 |
17
16
15 |
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0

Number taking exam: 372
Median: 75
Mean: 72.0
Standard Deviation: 16.5

Number receiving A's: 50 13.4%
Number receiving B's: 87 23.4
Number receiving C's: 93 25.0
Number receiving D's: 63 16.9
Number receiving F's: 79 21.2