Mathematics 204

Summer 2013

Exam II

Your Printed Name: Solution key

Your Instructor's Name: Sabrina Shrepel

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page, 6 pages of problems containing 6 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

problem	1	2	3	4	5	6	**Sum**
points earned							
maximum points	16	17	17	17	17	16	100

1.[16] Find the general solution to the differential equation $y^{(4)} - 3y'' - 4y = 0$.

$y = e^{rt}$

$r^4 - 3r^2 - 4 = 0$ $\boxed{r^2 = u}$

$u^2 - 3u - 4 = 0$

$(u-4)(u+1) = 0$

$u_1 = 4$ $u_2 = -1$

$r^2 = u$ $r^2 = u$

$r = \pm\sqrt{u}$ $r = \pm\sqrt{u}$

$r_1 = 2$ $r_3 = i$

$r_2 = -2$ $r_4 = -i$

$\Rightarrow$ $y_1 = e^{2t}$ $y_2 = e^{-2t}$ $y_3 = \cos t$ $y_4 = \sin t$

$\Rightarrow$ $y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos t + c_4 \sin t$

2.[17] Find a particular solution of $t^2y''-3ty'+4y=t^2\ln t$ for t>0. Use the fact that $y_1=t^2, y_2=t^2\ln t$ are linearly independent solutions of the corresponding homogeneous differential equation.

Use VoP-method to obtain $y_p = u_1(t)y_1 + u_2(t)y_2$

with $y_1(t)=t^2$, $y_2(t)=t^2\ln t$.

Use:

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt$$

with $g(t) = \frac{t^2\ln t}{t^2} = \ln t$ — RHS in Stand. Form

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt$$

$$W(y_1,y_2)(t) = \begin{vmatrix} t^2 & t^2\ln t \\ 2t & 2t\ln t + t^2\cdot\frac{1}{t} \end{vmatrix} = t^3 2\ln t + t^3 - 2t^3\ln t = t^3 \neq 0 \text{ for } t>0$$

$$u_1(t) = -\int \frac{t^2\ln t\cdot\ln t}{t^3}\,dt = -\int \frac{\ln^2 t}{t}\,dt \overset{\substack{u=\ln t\\ du=\frac{1}{t}dt}}{=} -\int u^2\,du = -\frac{1}{3}u^3 + \cancel{A}^{0} =$$

$$= -\frac{1}{3}u^3 = -\frac{1}{3}[\ln t]^3$$

$$u_2(t) = \int \frac{t^2\ln t}{t^3}\,dt = \int \ln t\cdot\frac{1}{t}\,dt \overset{\substack{u=\ln t\\ du=\frac{1}{t}dt}}{=} \int u\,du = \frac{1}{2}u^2 + \cancel{A}^{0} = \frac{1}{2}[\ln t]^2$$

$$\Rightarrow y_p(t) = -\frac{1}{3}(\ln t)^3\cdot t^2 + \frac{1}{2}(\ln t)^2\cdot t^2\ln t = -\frac{1}{3}t^2(\ln t)^3 + \frac{1}{2}t^2(\ln t)^3 =$$

$$= \frac{1}{6}t^2(\ln t)^3$$

3.[17] Find the general solution to the differential equation $y''' - y'' - 2y' = e^{-t}$.

Step 1 Solve $y''' - y'' - 2y' = 0$

$y = e^{rt} \rightarrow r^3 - r^2 - 2r = 0 \rightarrow r(r^2 - r - 2) = 0$

$r_1 = 0 \quad r^2 - r - 2 = (r-2)(r+1) = 0 \rightarrow r_2 = 2 \quad r_3 = -1$

$y_h(t) = c_1 e^{0 \cdot t} + c_2 e^{2t} + c_3 e^{-t} = c_1 + c_2 e^{2t} + c_3 e^{-t}$

Step 2 Use MUC to obtain $y_p(t)$

$y_p(t) = t^s \cdot A e^{-t} \overset{s=1}{=} t \cdot A \cdot e^{-t} \rightarrow y_p' = Ae^{-t}[1-t]$

$y_p'' = Ae^{-t}[t-2]$

$y_p''' = Ae^{-t}[3-t]$

$\Rightarrow Ae^{-t}[3-t] - Ae^{-t}[t-2] - 2Ae^{-t}[1-t] \overset{!}{=} e^{-t}$

$A(3-t) - A(t-2) - 2A(1-t) \overset{!}{=} 1$

t-terms $-A - A + 2A = 0$ ✓

c-terms $3A + 2A - 2A \overset{!}{=} 1 \Rightarrow 3A \overset{!}{=} 1 \Rightarrow A = 1/3$

$\Rightarrow y_p(t) = \frac{1}{3} t e^{-t}$

$\Rightarrow y(t) = c_1 + c_2 e^{2t} + c_3 e^{-t} + \frac{1}{3} t e^{-t}$

4.[17] [In the following problem, assume that the acceleration of gravity is 9.8 meters per second per second.] A 2 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 10 centimeters beyond its natural length. The body is acted upon by a downward external force of $10\sin(t/2)$ newtons. Assume that no damping forces act on the body.

(a) If the body is set in motion from its equilibrium position with an upward initial velocity of 20 centimeters per second, set up, **BUT DO NOT SOLVE,** an initial value problem that describes the motion of the body.

$mu'' + \gamma u' + ku = F(t)$

$2u'' + 196u = 10\sin(t/2)$

$u(0) = 0 \qquad u'(0) = -0.2$

[upwards]

$\gamma = 0 \quad F(t) = 10\sin(t/2)$

$m = 2[kg] \quad \boxed{k \cdot u_0 = m \cdot g}$ cm in m

$\Rightarrow k \cdot 0.1 = 2 \cdot 9.8$

$k = 20 \cdot 9.8 = 196$

(b) Considering the model above replacing the external force by $4\cos(\omega t)$ newtons, find the value of the frequency ω which will cause resonance or explain why there is no such frequency.

$2u'' + 196u = 4\cos(\omega t)$

If driver's frequency equal to natural frequency then resonance occurs (for no damping or small damping).

$\Rightarrow$ Solve for natural frequ.: $2u'' + 196u = 0 \quad \boxed{u = e^{rt}}$

$2r^2 + 196 = 0 \qquad r = \pm\sqrt{\frac{-196}{2}} = \pm\sqrt{-98} =$

$= \pm i\sqrt{98}$

$(= \pm i\, 7\sqrt{2})$

$\Rightarrow$ resonance occurs if

$\omega = \sqrt{98} = 7\sqrt{2}$

5.[17] Find the general solution to the following differential equation $2t^2y''+ty'-3y=0$ on the interval t>0.

(Euler DE) $y(t)=t^r$

$$2t^2r(r-1)t^{r-2}+trt^{r-1}-3t^r=0$$

$$2r(r-1)+r-3=0$$

$$2r^2-2r+r-3=0$$

$$2r^2-r-3=0$$

$$r_{1,2}=\frac{1\pm\sqrt{1-4\cdot2\cdot(-3)}}{4}=\frac{1\pm\sqrt{25}}{4}=\frac{1\pm5}{4}$$

$$r_1=\frac{6}{4}=\frac{3}{2} \qquad r_2=\frac{-4}{4}=-1$$

$$\Rightarrow y_1(t)=t^{3/2} \qquad y_2(t)=t^{-1}$$

$$\Rightarrow y(t)=c_1t^{3/2}+c_2t^{-1}$$

6.[16] Use the definition of the Laplace transform,

$$\mathcal{L}\{f\}(s)=\int_0^\infty f(t)e^{-st}dt$$

for those values of s for which the improper integral converges, to find the Laplace transform of the function
$f(t)=\begin{cases}1 & 0\le t\le 1\\ e^t & t>1\end{cases}$.
For which values of s is the Laplace transform of f defined?

$$\mathcal{L}\{f\}(s)=\int_0^\infty f(t)e^{-st}\,dt=\int_0^1 1\cdot e^{-st}\,dt+\int_1^\infty e^t e^{-st}\,dt=$$

$$=-\frac{1}{s}e^{-st}\Big|_0^1+\lim_{n\to\infty}\int_1^n e^{t(1-s)}\,dt=$$

$$=-\frac{1}{s}e^{-s}-\Big(-\frac{1}{s}\underbrace{e^{-s\cdot 0}}_{=1}\Big)+\lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\Big|_1^n\right]=$$

$$=-\frac{e^{-s}}{s}+\frac{1}{s}+\underbrace{\lim_{n\to\infty}\left[\frac{1}{1-s}e^{n(1-s)}\right]}_{\to 0 \text{ for } \boxed{s>1}}-\frac{1}{1-s}\cdot e^{1(1-s)}=$$

$$=\frac{1}{s}-\frac{e^{-s}}{s}-\frac{1}{1-s}e^{(1-s)}$$

provided $s>1$.

2013 Summer Semester, Math 204 Hour Exam _2___
Instructor __Sabrina Streipert___, Section ___A__

100	2	59	2	19
99		58		18
98	2	57	1	17
97		56	2	16
96	2	55		15
95	2	54		14
94	1	53		13
93	2	52		12
92	1	51		11
91		50		10
90	3	49		9
89	3	48		8
88	1	47		7
87	1	46		6
86		45	1	5
85		44		4
84	1	43		3
83	2	42		2
82		41		1
81	1	40		0
80	1	39		
79		38		
78	2	37		
77		36		
76	3	35		
75		34		
74	2	33		
73	1	32		
72		31		
71	1	30		
70	3	29		
69	2	28		
68	1	27	1	
67	1	26		
66		25		
65		24		
64	2	23		
63		22		
62	1	21		
61	2	20		
60				

Number taking exam:_53__
Median: __78.0___
Mean: __77.0____
Standard Deviation: 15.6___

Number receiving A's:____15___ (28.3%)
Number receiving B's:____10__ (18.9%)
Number receiving C's:_____12__ (22.6%)
Number receiving D's:____9___ (17.0%)
Number receiving F's:____7___ (13.2%)