Mathematics 204

Fall 2013

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. Express all solutions in real-valued, simplified form.
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	20	18	20	20	22	100

1.[20] Solve the initial value problem $y'' + 2y' - 3y = \delta(t-3)$, $y(0) = 0$, $y'(0) = 1$.

Because of the presence of the Dirac delta in the DE, we use the Laplace transform method:

$$\mathcal{L}\{y'' + 2y' - 3y\}(s) = \mathcal{L}\{\delta(t-3)\}(s).$$

Linearity of the transform and formulas 6 and 9 in the table of Laplace transforms yield

$$s^2\mathcal{L}\{y\}(s) - \cancel{sy(0)}^{\,0} - y'(0)^{1} + 2\left(s\mathcal{L}\{y\}(s) - \cancel{y(0)}^{\,0}\right) - 3\mathcal{L}\{y\}(s) = e^{-3s}.$$

Applying the initial conditions and rearranging gives

$$(s^2 + 2s - 3)\mathcal{L}\{y\}(s) = 1 + e^{-3s}$$

or

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2+2s-3} + e^{-3s}\cdot\frac{1}{s^2+2s-3}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-3}\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2+2s-3}\right\}. \qquad (*)$$

Using a partial fraction decomposition,

$$\frac{1}{s^2+2s-3} = \frac{1}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1} \Rightarrow 1 = A(s-1) + B(s+3).$$

Setting $s=1$ yields $1 = A(0) + B(4)$ so $B = 1/4$. Setting $s=-3$ yields $1 = A(-4) + B(0)$ so $A = -1/4$. Therefore, using formula 1 in the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1/4}{s-1} - \frac{1/4}{s+3}\right\} = \frac{1}{4}e^{t} - \frac{1}{4}e^{-3t}.$$

Using formula 8 in the table and the previous work,

$$\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2+2s-3}\right\} = u_3(t)f(t-3) \quad \text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-3}\right\} = \frac{1}{4}e^{t} - \frac{1}{4}e^{-3t}.$$

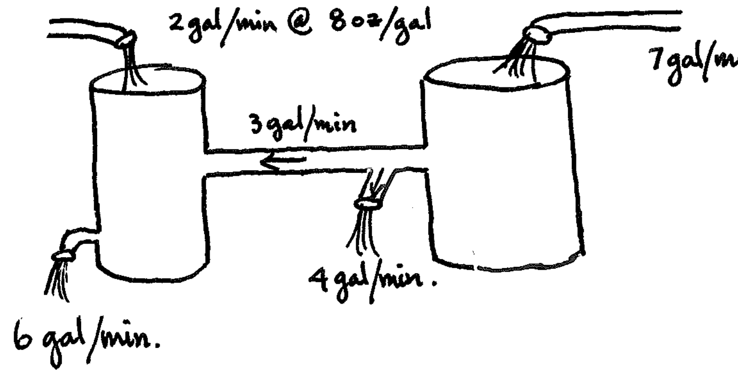
Thus $\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2+2s-3}\right\} = u_3(t)\frac{1}{4}\left(e^{t-3} - e^{-3(t-3)}\right)$. Substituting in (*) gives

$$\boxed{y(t) = \frac{1}{4}e^{t} - \frac{1}{4}e^{-3t} + u_3(t)\frac{1}{4}\left(e^{t-3} - e^{-3(t-3)}\right).}$$

2. Two water tanks initially hold 100 gallons of pure water each. A mixture of salt and water at a concentration of 5 ounces per gallon flows into tank 2 at a rate of 7 gallons per minute. A mixture of salt and water at a concentra-tion of 8 ounces per gallon flows into tank 1 at a rate of 2 gallons per minute. The well-stirred mixture drains from tank 2 into tank 1 at a rate of 3 gallons per minute and into the environment at a rate of 4 gallons per minute. The well-stirred mixture in tank 1 also drains into the environment at a rate of 6 gallons per minute.

(a) [16] Set up, **BUT DO NOT SOLVE**, an initial value problem modeling the number of ounces $Q_1(t)$ and $Q_2(t)$ of salt in tanks 1 and 2, respectively, at time t minutes.

(b) [2] For which times t is your model in part (a) valid? Explain why this is so.



Notes: The volume of solution in tank 2 is constant so $V_2(t) = V_2(0) = 100$ gallons. The volume of solution in tank 1 obeys $\frac{dV_1}{dt} = \frac{3\,gal}{min} + \frac{2\,gal}{min} - 6\frac{gal}{min} = -1\frac{gal}{min}$.

Therefore $V_1(t) = V_1(0) - t$ gallons $= (100-t)$ gallons.

(a) We apply the principle, Net rate of change of salt with time = Rate of inflow of salt with time − Rate of outflow of salt with time, to each tank:

Tank 1:
$$\frac{dQ_1}{dt} = \left(\frac{2\,gal}{min}\right)\left(\frac{8\,oz}{gal}\right) + \left(\frac{3\,gal}{min}\right)\left(\frac{Q_2(t)\,oz}{100\,gal}\right) - \left(\frac{6\,gal}{min}\right)\left(\frac{Q_1(t)\,oz}{(100-t)\,gal}\right).$$

Tank 2:
$$\frac{dQ_2}{dt} = \left(\frac{7\,gal}{min}\right)\left(\frac{5\,oz}{gal}\right) - \left(\frac{7\,gal}{min}\right)\left(\frac{Q_2(t)\,oz}{100\,gal}\right).$$

Simplifying and adjoining the initial conditions yields the IVP

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= \left(\frac{-6}{100-t}\right)Q_1 + \left(\frac{3}{100}\right)Q_2 + 16, \quad & Q_1(0) = 0, \\ \frac{dQ_2}{dt} &= -\frac{7}{100}Q_2 + 35, \quad & Q_2(0) = 0, \end{aligned}}$$

where Q_1 and Q_2 are in ounces and t is in minutes.

(b) Since Tank 1 is empty when $t = 100$ minutes, the model is valid over the time interval $0 \le t < 100$. Furthermore, $V_1(t) = 100 - t < 0$ if $t > 100$, so this is impossible as well.

3. (a) [6] Calculate the Wronskian of the vector functions $\mathbf{x}^{(1)}(t)=\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ and $\mathbf{x}^{(2)}(t)=\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$.

(b) [14] Find a constant matrix $\mathbf{A}$ such that both vector functions in part (a) are solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) $W(\vec{x}^{(1)}, \vec{x}^{(2)})(t) = \det\left[\vec{x}^{(1)}(t) \;\; \vec{x}^{(2)}(t)\right] = \det\begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} = 2e^{2t} - (-2e^{2t}) = \boxed{4e^{2t}}$.

(b) Let A be a constant 2x2 matrix such that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ solve $\vec{x}' = A\vec{x}$. Then

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} = \vec{x}^{(1)\prime}(t) = A\vec{x}^{(1)}(t) = A\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \quad \text{and} \quad \begin{bmatrix} 3 \\ 6 \end{bmatrix} e^{3t} = \vec{x}^{(2)\prime}(t) = A\vec{x}^{(2)}(t) = A\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \text{ for all } t.$$

Cancelling the exponential functions in each vector/matrix equation gives the algebraic system $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If we write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then this is equivalent to the system of scalar equations:

$$\begin{cases} a - 2b = -1 \\ c - 2d = 2 \\ a + 2b = 3 \\ c + 2d = 6. \end{cases}$$

Adding equations 1 and 3 yields $2a = 2$ so $a = 1$. Adding equations 2 and 4 yields $2c = 8$ so $c = 4$. Substituting $a = 1$ in equation 1 yields $1 - 2b = -1$ so $b = 1$. Substituting $c = 4$ in equation 2 yields $4 - 2d = 2$ so $d = 1$.

Therefore $\boxed{A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.}$

4. [20] Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} -4 & 3 \\ -3 & 6 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$.

Let $A = \begin{bmatrix} -4 & 3 \\ -3 & 6 \end{bmatrix}$. Then $\vec{x} = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$. To find the eigenvalues r of A we solve $0 = \det(A - rI) = \begin{vmatrix} -4-r & 3 \\ -3 & 6-r \end{vmatrix} = (r-6)(r+4) - (-3)(3)$ or

$0 = r^2 - 2r - 15 = (r-5)(r+3)$.

Eigenvalues	Eigenvectors
$r_1 = 5$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
$r_2 = -3$	$\vec{k}^{(2)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The eigenvectors of A satisfy $(A - rI)\vec{k} = \vec{0}$. When $r = r_1 = 5$ this is $\begin{bmatrix} -4-5 & 3 \\ -3 & 6-5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently

$\begin{cases} \cancel{-9k_1 + 3k_2 = 0} & \text{Redundant; it is 3 times the second equation.} \\ -3k_1 + k_2 = 0 & \Rightarrow k_2 = 3k_1. \end{cases}$

$\therefore\ \vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 3k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

When $r = r_2 = -3$, $(A - rI)\vec{k} = \vec{0}$ becomes $\begin{bmatrix} -4-(-3) & 3 \\ -3 & 6-(-3) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is equivalent to

$\begin{cases} -1k_1 + 3k_2 = 0 & \Rightarrow 3k_2 = k_1. \\ \cancel{-3k_1 + 9k_2 = 0} & \text{Redundant; it is 3 times the first equation.} \end{cases}$

$\therefore\ \vec{k}^{(2)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Therefore $\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{r_1 t} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{5t}$ and $\vec{x}^{(2)}(t) = \vec{k}^{(2)}e^{r_2 t} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{-3t}$ solve $\vec{x}' = A\vec{x}$.

Since $W(\vec{x}^{(1)}, \vec{x}^{(2)})(t) = \det\begin{bmatrix} e^{5t} & 3e^{-3t} \\ 3e^{5t} & e^{-3t} \end{bmatrix} = -8e^{2t} \neq 0$, they form a F.S.S. and

$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{-3t}$ is the general solution of $\vec{x}' = A\vec{x}$. We want

$$\begin{bmatrix} 4 \\ -4 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ so } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1-9}\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -4 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 16 \\ -16 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$ Thus

$$\boxed{\vec{x}(t) = -2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{5t} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{-3t}}$$ solves the IVP.

5. (a) [20] Solve the initial value problem $y''+y=g(t)$, $y(0)=0$, $y'(0)=1$, where

$$g(t)=\begin{cases}0 & \text{if } 0\le t<1,\\ t-1 & \text{if } 1\le t.\end{cases}$$

(b) [2] Which is greater, $y\left(\frac{\pi}{4}\right)$ or $y\left(1+\frac{\pi}{4}\right)$? Justify your answer.

(a) Using the unit step function, the DE may be written $y''+y=u_1(t)(t-1)$. Applying the Laplace transform method and formula 2 in the table of Laplace transforms we find

$$s^2\mathcal{L}\{y\}(s)-s\overset{0}{y(0)}-\overset{1}{y'(0)}+\mathcal{L}\{y\}(s)=\mathcal{L}\{y''+y\}(s)=\mathcal{L}\{u_1(t)(t-1)\}(s)=e^{-s}\mathcal{L}\{t\}(s)=\frac{e^{-s}}{s^2}.$$

Applying the initial conditions and rearranging we find

$$(s^2+1)\mathcal{L}\{y\}(s)=1+\frac{e^{-s}}{s^2} \quad\text{or}\quad y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}+e^{-s}\cdot\frac{1}{s^2(s^2+1)}\right\}.$$

Using a partial fraction decomposition we find

$$\frac{1}{s^2(s^2+1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+1} \quad\text{so}\quad 1=As(s^2+1)+B(s^2+1)+(Cs+D)s^2.$$

Setting $s=0$ yields $1=A(0)+B(1)+(C(0)+D)0^2 \Rightarrow B=1$.

Setting $s=i$ yields $1=A(0)+B(0)+(Ci+D)(-1) \Rightarrow 1=-D-Ci$ so $D=-1$ and $C=0$.

Setting $s=1$ yields $1=A(2)+B(2)+(C+D)1^2 \Rightarrow 1=2A+2(1)+(0+-1) \Rightarrow A=0$.

Thus, table entries 3, 8, and 2 give

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{e^{-s}\cdot\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)\right\}=\sin(t)+u_1(t)\cdot f(t-1)$$

where $f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\}=t-\sin(t)$. Thus $\boxed{y(t)=\sin(t)+u_1(t)\left[t-1-\sin(t-1)\right]}$.

(b) Since $\frac{\pi}{4}<1$ and $1+\frac{\pi}{4}>1$, $u_1(\frac{\pi}{4})=0$ and $u_1(1+\frac{\pi}{4})=1$. Thus $y(\frac{\pi}{4})=\sin(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$ and

$$y\left(1+\frac{\pi}{4}\right)=\sin\left(1+\frac{\pi}{4}\right)+\left[1+\frac{\pi}{4}-1-\sin\left(1+\frac{\pi}{4}-1\right)\right]=\sin(1)\cos\left(\frac{\pi}{4}\right)+\cos(1)\sin\left(\frac{\pi}{4}\right)+\frac{\pi}{4}-\sin\left(\frac{\pi}{4}\right)$$

$$=(\sin(1)+\cos(1))\frac{\sqrt{2}}{2}+\frac{\pi}{4}-\frac{\sqrt{2}}{2}>(\sin^2(1)+\cos^2(1))\frac{\sqrt{2}}{2}+\frac{\pi}{4}-\frac{\sqrt{2}}{2}=\frac{\pi}{4}.$$

Since $\frac{\pi}{4}>\frac{3.14}{4}=0.785$ and $\frac{\sqrt{2}}{2}\doteq 0.707$ it follows that $\boxed{y\left(1+\frac{\pi}{4}\right)>y\left(\frac{\pi}{4}\right)}$.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$
7. $e^{ct}f(t)$	$F(s-c)$
8. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
9. $\delta(t-c)$	e^{-cs}

2013 Fall Semester, Math 204 Hour Exam III, Master List

Score	Tally	Score	Tally	Score	Tally
100	|||	59	|||	19	
99	卌 ||||	58	卌	18	|
98	||||	57	卌 |||	17	
97	卌 ||||	56	||	16	
96	卌 |	55	||||	15	
95	卌 ||	54	||	14	
94	卌	53	卌	13	|
93	卌 卌 |	52	|	12	||
92	卌 |||	51	|	11	
91	卌 卌 卌	50	||	10	
90	卌	49	|	9	
89	卌 ||||	48	||	8	
88	卌 |||	47		7	
87	卌 ||	46	|	6	
86	卌 卌 |	45	||	5	
85	卌 卌 |||	44		4	
84	卌 卌 卌	43	|||	3	|
83	卌 卌 卌 卌 |	42		2	
82	卌 卌 ||	41		1	
81	卌 卌 |||	40	|||	0	||
80	卌 卌 |||	39	|		
79	卌 卌 卌 |	38			
78	卌 卌 |	37			
77	卌 卌 卌 ||	36	||		
76	卌 卌	35	|		
75	卌 卌 ||||	34			
74	卌	33	|		
73	卌 卌 ||||	32	|		
72	卌 卌 ||||	31			
71	卌 |||	30			
70	卌 卌 卌 |	29			
69	卌 卌	28			
68	卌 |	27			
67	卌	26			
66	||||	25			
65	卌 ||||	24			
64	|||	23			
63	卌 |	22			
62		21			
61	||||	20			
60	卌 ||				

82 As ✓

122 Bs ✓

125 Cs ✓

54 Ds ✓

58 Fs ✓

Number taking exam: 441 ✓
Median: 78
Mean: 75.9
Standard Deviation: 16.1

Number receiving A's: 82 — 18.6%
Number receiving B's: 122 — 27.7
Number receiving C's: 125 — 28.3
Number receiving D's: 54 — 12.2
Number receiving F's: 58 — 13.2

2013 Fall Semester, Math 204 Hour Exam III

Instructor Grow, Section D

100			59			19
99	II		58			18
98	I		57			17
97	I		56			16
96			55		1 F	15
95		6 As	54			14
94			53			13
93			52	I		12
92	I		51			11
91	I		50			10
90			49			9
89	II		48			8
88			47			7
87	II		46			6
86	I		45			5
85		9 Bs	44			4
84	II		43			3
83			42			2
82			41			1
81	II		40			0
80			39			
79	II		38			
78	II		37			
77	I		36			
76	I		35			
75		12 Cs	34			
74			33			
73			32			
72	III		31			
71			30			
70	III		29			
69	I		28			
68			27			
67			26			
66			25			
65	II	5 Ds	24			
64			23			
63	I		22			
62			21			
61			20			
60	I					

Number taking exam: 33
Median: 79
Mean: 79.1
Standard Deviation: 11.6

Number receiving A's: 6 — 18.2%
Number receiving B's: 9 — 27.3
Number receiving C's: 12 — 36.4
Number receiving D's: 5 — 15.2
Number receiving F's: 1 — 3.0