Mathematics 204

Spring 2013

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. You may find useful the facts $\Gamma(1/2)=\sqrt{\pi}/2$, $\Gamma(p+1)=p\Gamma(p)$ if p is a positive real number, and $\Gamma(n+1)=n!$ if n is a nonnegative integer.
6. Once the exam begins, you will have 60 minutes to complete your solutions.
7. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
8. Express all solutions in real-valued, simplified form.
9. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
10. The symbol [18] at the beginning of a problem indicates the point value of that problem is 18. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	18	22	18	20	22	100

1.[18] Compute the inverse Laplace transform of $F(s)=\dfrac{2s^2+6s+12}{s^3-4s}$.

Partial Fraction Decomposition

$$F(s)=\frac{2s^2+6s+12}{s(s-2)(s+2)} \overset{\checkmark}{=} \frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+2}.$$

$$\therefore\quad 2s^2+6s+12=\left(\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+2}\right)s(s-2)(s+2)$$

$$\Rightarrow\quad 2s^2+6s+12=A(s-2)(s+2)+Bs(s+2)+Cs(s-2)$$

To find A, set $s=0$: $\quad 12=-4A \quad$ so $\quad A=-3.$

To find B, set $s=2$: $\quad 8+12+12=8B \quad$ so $\quad B=4.$

To find C, set $s=-2$: $\quad 8-12+12=8C \quad$ so $\quad C=1.$

Therefore

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{-3}{s}+\frac{4}{s-2}+\frac{1}{s+2}\right\}=\boxed{-3+4e^{2t}+e^{-2t}}.$$

(Here we ~~have~~ used linearity of the inverse Laplace transform together with formula 1 (with $a=0$, $a=2$, and $a=-2$, respectively) in the Laplace transform table.)

2. (a) [20] Solve the initial value problem $y'' + 4y = g(t)$, $y(0) = 1$, $y'(0) = 0$, where

$$g(t) = \begin{cases} 0 & \text{if } 0 \le t < 3\pi, \\ 1 & \text{if } 3\pi \le t < \infty. \end{cases}$$

(b) [2] Which is greater, $y(2\pi)$ or $y(5\pi)$? Justify your answer.

(a) Note that $g(t) = u_{3\pi}(t)$, the unit step function. Substituting this for g and taking the Laplace transform of the resulting DE gives

$$\mathcal{L}\{y'' + 4y\}(s) = \mathcal{L}\{u_{3\pi}\}(s).$$

Linearity of the Laplace transform, together with formulas 6 (with $n=2$) and 8 (with $c = 3\pi$ and $f(t) \equiv 1$) in the Laplace transform table, yield

$$s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + 4\mathcal{L}\{y\}(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions $y(0) = 1$ and $y'(0) = 0$, and rearranging produces

$$(s^2 + 4)\mathcal{L}\{y\}(s) = s + \frac{e^{-3\pi s}}{s}$$

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+4} + e^{-3\pi s} \cdot \frac{1}{s(s^2+4)}.$$

A straightforward partial fraction decomposition calculation leads to

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad \text{where } A = \frac{1}{4},\ B = -\frac{1}{4}, \text{ and } C = 0.$$

Thus
$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} + e^{-3\pi s} \cdot \left[\frac{1/4}{s} + \frac{-1/4\, s}{s^2+4} \right] \right\}$$

or
$$\boxed{y(t) = \cos(2t) + u_{3\pi}(t)\left[\frac{1}{4} - \frac{1}{4}\cos(2(t-3\pi)) \right]}$$

by linearity of the Laplace transform and table entries 4 and 8.

(b) $y(2\pi) = \cos(4\pi) + 0 = 1$

$y(5\pi) = \cos(10\pi) + 1 \cdot \left[\frac{1}{4} - \frac{1}{4}\cos(4\pi) \right] = 1$

Therefore, neither is greater than the other: $y(2\pi) = 1 = y(5\pi)$.

3.[18] Let $\mathbf{A}(t)=\begin{pmatrix} \sin(t) & -\cos(t) \\ -\cos(t) & -\sin(t) \end{pmatrix}$. Does $\frac{d}{dt}\mathbf{A}^2$ equal $2\mathbf{A}\,d\mathbf{A}/dt$ or $2\,d\mathbf{A}/dt\,\mathbf{A}$ or $\mathbf{A}\,d\mathbf{A}/dt + d\mathbf{A}/dt\,\mathbf{A}$ or none of these? Show work supporting your answer.

Solution 1: The product rule for matrix functions is

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

Taking $B = A$ yields

$$\boxed{\frac{d}{dt}A^2 = A\frac{dA}{dt} + \frac{dA}{dt}A}.$$

Solution 2:

$$A^2 = \begin{bmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{bmatrix}\begin{bmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{bmatrix} = \begin{bmatrix} \sin^2 t + \cos^2 t & -\sin t\cos t + \cos t\sin t \\ -\cos t\sin t + \sin t\cos t & \cos^2 t + \sin^2 t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\frac{d}{dt}A^2 = \frac{d}{dt}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. On the other hand,

$$A\frac{dA}{dt} + \frac{dA}{dt}A = \begin{bmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} + \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}\begin{bmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{bmatrix}$$

$$= \begin{bmatrix} \sin t\cos t - \cos t\sin t & \sin^2 t + \cos^2 t \\ -\cos^2 t - \sin^2 t & -\cos t\sin t + \sin t\cos t \end{bmatrix} + \begin{bmatrix} \cos t\sin t - \sin t\cos t & -\cos^2 t - \sin^2 t \\ \sin^2 t + \cos^2 t & -\sin t\cos t + \cos t\sin t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore $\boxed{\frac{d}{dt}A^2 = A\frac{dA}{dt} + \frac{dA}{dt}A}$.

The above calculations show $2A\frac{dA}{dt} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $2\frac{dA}{dt}A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, so $\frac{d}{dt}A^2$ is not equal to either $2A\frac{dA}{dt}$ or $2\frac{dA}{dt}A$.

4.[20] Solve the integral equation $1 = \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau$ for $y(t)$.

Recall the definition of the convolution product: $(g*f)(t) = \int_0^t g(\tau)f(t-\tau)d\tau$.
Therefore the integral equation can be written as

4 pts. to here

$$1 = (y*f)(t) \quad (+2)$$

where $f(t) = \frac{1}{\sqrt{t}} = t^{-1/2}$. (+2) Taking the Laplace transform of the integral equation yields

$$\frac{1}{s} = \mathcal{L}\{1\}(s) = \mathcal{L}\{y*f\}(s) = \mathcal{L}\{y\}(s)\mathcal{L}\{f\}(s)$$

(+3) (+3) 10 pts. to here.

by entries 1 (with a=0) and 5 in the table of Laplace transforms.

But $\mathcal{L}\{f\}(s) = \mathcal{L}\{t^{-1/2}\}(s) = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}/2}{s^{1/2}}$ (+3) by entry 2 (with $p=-1/2$) in the table and one of the facts in 5 on the front page of this exam. Thus

13 pts. to here.

$$\frac{1}{s} = \mathcal{L}\{y\}(s) \cdot \frac{\sqrt{\pi}/2}{s^{1/2}} \quad \text{so} \quad \frac{2}{\sqrt{\pi}} \cdot \frac{1}{s^{1/2}} = \mathcal{L}\{y\}(s).$$

(+3) 16 pts. to here.

Taking the inverse Laplace transform and using entry 2 (with $p=-1/2$) in the table gives

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{\pi}} \cdot \frac{1}{s^{1/2}}\right\} = \frac{2}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{2}{\sqrt{\pi}} \cdot \frac{t^{-1/2}}{\Gamma(-\frac{1}{2}+1)} = \frac{2}{\sqrt{\pi}} \cdot \frac{t^{-1/2}}{\Gamma(1/2)}.$$

(+3) 19 pts. to here.

Consequently, since $\Gamma(1/2) = \sqrt{\pi}/2$, we have

$$\boxed{y(t) = \frac{4}{\pi\sqrt{t}}}.$$

(+1) 20 pts. to here

5.(a) [20] Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) [2] Describe the behavior of the solution as $t \to \infty$.

(a) Let $A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$. Then $\vec{x} = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$, the eigenvalue equation for the matrix A. The eigenvalues r of A satisfy

$$0 = \det(A - rI) = \begin{vmatrix} 4-r & -3 \\ 8 & -6-r \end{vmatrix} = (r+6)(r-4) + 24 = r^2 + 2r = r(r+2).$$

Eigenvalues	Eigenvectors
$r_1 = 0$	$\vec{k}^{(1)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
$r_2 = -2$	$\vec{k}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

To find eigenvectors of A corresponding to $r=0$, we solve $(A - rI)\vec{k} = 0$ when $r=0$. That is,

$$\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or equivalently } \begin{cases} 4k_1 - 3k_2 = 0 \\ \cancel{8k_1 - 6k_2 = 0}. \end{cases}$$

But the second equation is redundant, being twice the first equation. Thus $4k_1 - 3k_2 = 0 \Rightarrow k_2 = \frac{4}{3}k_1$.

$$\therefore \vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ \frac{4}{3}k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 4/3 \end{bmatrix}.$$ Take $k_1 = 3$ to get $\vec{k}^{(1)}$.

To find eigenvectors of A corresponding to $r=-2$, we solve $(A - rI)\vec{k} = \vec{0}$ when $r=-2$. I.e.

$$\begin{bmatrix} 4-(-2) & -3 \\ 8 & -6-(-2) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or equivalently } \begin{cases} 6k_1 - 3k_2 = 0 \\ \cancel{8k_1 - 4k_2 = 0}. \end{cases}$$

But the second equation is redundant, being $\frac{4}{3}$ times the first equation. Therefore $6k_1 - 3k_2 = 0$ implies $k_2 = 2k_1$ so $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 2k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Take $k_1 = 1$ to get $\vec{k}^{(2)}$. It follows that

$\vec{x}^{(1)} = \vec{k}^{(1)}e^{r_1 t} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{x}^{(2)} = \vec{k}^{(2)}e^{r_2 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-2t}$ are solutions to $\vec{x}' = A\vec{x}$. However

$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} 3 & e^{-2t} \\ 4 & 2e^{-2t} \end{vmatrix} = 2e^{-2t} \neq 0$, so $\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)$ form a fundamental set of solutions to $\vec{x}' = A\vec{x}$ and hence $\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) = c_1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-2t}$ is the general solution. Applying the initial condition gives $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so $\boxed{\vec{x}(t) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-2t}.}$

(b) Since $e^{-2t} \to 0$ as $t \to \infty$, $\boxed{\vec{x}(t) \longrightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ as } t \to \infty}$.

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^p	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $(f * g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}