Mathematics 204

Spring 2013

Final Exam

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 10 pages of problems containing 10 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown for all integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

problem	1	2	3	4	5	6	7	8	9	10	Sum
points earned											
maximum points	20	21	21	21	14	20	21	20	21	21	200

1.[20] Find the explicit solution of the initial value problem $y' = y(1-y)$, $y(0) = 0.01$.

The DE is first-order and autonomous, i.e. of the form $y' = f(y)$. Hence it is a variables separable DE. Writing the DE as

$$\frac{dy}{dt} = y(1-y)$$

and separating variables yields

$$\frac{dy}{y(1-y)} = dt.$$

P.F.D.

$$\frac{1}{y(1-y)} \stackrel{\text{P.F.D.}}{=} \frac{A}{y} + \frac{B}{1-y}$$

$$\Rightarrow \quad 1 = A(1-y) + By$$

Set $y=0$: $1 = A$

Set $y=1$: $1 = B$

Integrating both sides and using the partial fraction decomposition above yields

$$\int \frac{1}{y(1-y)}\,dy = \int dt$$

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = t + c$$

$$\ln|y| - \ln|1-y| = t + c$$

$$\ln\left|\frac{y}{1-y}\right| = t + c$$

$$\left|\frac{y}{1-y}\right| = e^{t+c}$$

$$\frac{y}{1-y} = Ke^{t}, \qquad (K = \pm e^{c}).$$

Using the initial condition gives $\frac{y(0)}{1-y(0)} = Ke^{0}$ so $\frac{0.01}{1-0.01} = K$ and $K = \frac{0.01}{0.99} = \frac{1}{99}$. Thus $\frac{y}{1-y} = \frac{1}{99}e^{t}$. We solve for y by first cross-multiplying: $99y = (1-y)e^{t} = e^{t} - ye^{t}$. Thus $99y + e^{t}y = e^{t}$ or $(99+e^{t})y = e^{t}$ so $\boxed{y(t) = \frac{e^{t}}{99+e^{t}}}$ or equivalently $\boxed{y(t) = \frac{1}{99e^{-t}+1}}$.

2. (a) [17] Solve the initial value problem $y'+y=1-t,\quad y(0)=y_0$.

(b) [4] Find the value of y_0 for which the solution in part (a) touches, but does not cross, the $t-$axis.

(a) The DE is first-order and linear. An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int 1 dt} = e^{t}$. Multiplying through the DE by the integrating factor yields

Product Rule for derivatives!

$$(e^{t}y)' = e^{t}y' + e^{t}y = (1-t)e^{t}.$$

Integrating both sides gives

$$e^{t}y = \int (e^{t}y)'dt = \int \underbrace{(1-t)}_{u}\underbrace{e^{t}dt}_{dv} = \underbrace{(1-t)}_{u}\underbrace{e^{t}}_{v} - \int \underbrace{e^{t}}_{v}\underbrace{(-dt)}_{du} = (1-t)e^{t} + e^{t} + c.$$

Thus $y(t) = \left[(1-t)e^{t} + e^{t} + c\right]e^{-t} = 2-t+ce^{-t}$ where c is an arbitrary constant. Applying the initial condition leads to

$$y_0 = y(0) = 2-0+ce^{-0} = 2+c$$

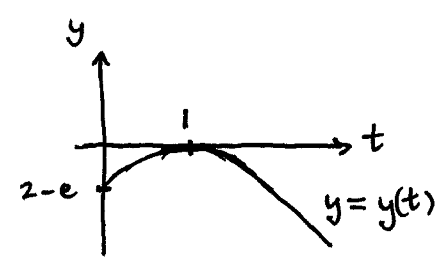
or equivalently $c = y_0 - 2$ so $\boxed{y(t) = 2-t+(y_0-2)e^{-t}}$ solves the I.V.P.

(b) Since the solution is to touch, but not cross, the t-axis, we must have $y(t_0) = 0 = y'(t_0)$ at some point t_0. Hence, substituting in the DE gives

$$0 = 0+0 = y'(t_0)+y(t_0) = 1-t_0$$

and it clearly follows that $t_0 = 1$. Substituting $t=1$ in the solution to the I.V.P. produces $0 = y(1) = 2-1+(y_0-2)e^{-1}$ so $(2-y_0)e^{-1} = 1$ and thus $2-y_0 = e$ or equivalently $\boxed{2-e = y_0}$. Substituting this expression for y_0 in the solution in part (a) to the I.V.P. leads to $y(t) = 2-t-e\cdot e^{-t}$ or $y(t) = 2-t-e^{1-t}$.

Note: One easily checks that $y(1) = 0 = y'(1)$.



3.[21] Solve $t^2y'' - 3ty' + 4y = t^2$ on the interval $t > 0$.

This is a nonhomogeneous, second-order Euler equation. If we assume a solution $y = t^m$ of the homogeneous Euler equation $t^2y'' - 3ty' + 4y = 0$ then $t^2 m(m-1)t^{m-2} - 3tmt^{m-1} + 4t^m = 0$ so $t^m[m(m-1) - 3m + 4] = 0$. But then $m^2 - 4m + 4 = 0$ so $(m-2)^2 = 0$ and $m = 2$ with multiplicity two. Therefore $y_c = c_1t^2 + c_2t^2\ln(t)$ is the general solution of $t^2y'' - 3ty' + 4y = 0$ on $t > 0$.

To obtain a particular solution of the nonhomogeneous equation $t^2y'' - 3ty' + 4y = t^2$ we use variation of parameters. Hence we must first normalize the DE, obtaining $y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 1$ so $g(t) = 1$. Then

$$W(y_1, y_2) = W(t^2, t^2\ln t) = \begin{vmatrix} t^2 & t^2\ln(t) \\ 2t & 2t\ln(t) + t^2 \cdot \frac{1}{t} \end{vmatrix} = 2t^3\ln(t) + t^3 - 2t^3\ln(t)$$

so $W(y_1, y_2) = t^3$. Thus, $y_p = u_1(t)y_1(t) + u_2(t)y_2(t) = t^2u_1(t) + t^2\ln(t)u_2(t)$ is a particular $\overset{\text{solution}}{}$ to $t^2y'' - 3ty' + 4y = t^2$ provided: Let $w = \ln(t)$. Then $dw = \frac{1}{t}dt$.

$$u_1(t) = \int -\frac{y_2 g}{W}\,dt = \int \frac{-t^2\ln(t) \cdot 1}{t^3}\,dt = \int -t^{-1} \cdot \ln(t)\,dt \overset{\text{*}}{=} \int -w\,dw$$

$$= -\frac{w^2}{2} + \cancel{c}^{\,0} = -\frac{1}{2}(\ln(t))^2$$

and

$$u_2(t) = \int \frac{y_1 g}{W}\,dt = \int \frac{t^2 \cdot 1}{t^3}\,dt = \int \frac{1}{t}\,dt = \ln(t) + \cancel{c}^{\,0}.$$

Therefore $y_p = t^2\left(\frac{-1}{2}\ln^2(t)\right) + t^2\ln(t) \cdot \ln(t) = \frac{t^2}{2}\ln^2(t)$. The general solution of $t^2y'' - 3ty' + 4y = t^2$ on $t > 0$ is

$$\boxed{y = y_c + y_p = c_1t^2 + c_2t^2\ln(t) + \frac{t^2}{2}\ln^2(t)}$$

where c_1 and c_2 are arbitrary constants.

$g_1(t) = 2t$ so $y_{p_1}(t) = t^s(A_1 t + A_0)$ where s is the multiplicity of 0 as a root of the characteristic equation; i.e. $s = 0$.

$g_2(t) = e^t$ so $y_{p_2}(t) = t^s B_0 e^t$ where s is the multiplicity of 1 as a root of the characteristic equation; i.e. $s = 0$.

4.[21] Find the general solution of $y^{(4)} - 16y = 2t + e^t$.

The DE is a linear, fourth-order, nonhomogeneous, constant coefficient equation. If we assume a solution $y = e^{rt}$ of the associated homogeneous DE, $y^{(4)} - 16y = 0$, then $r^4 - 16 = 0 \Rightarrow (r^2-4)(r^2+4) = 0 \Rightarrow$ $(r-2)(r+2)(r^2+4) = 0$ so $r = 2$ (multiplicity 1), $r = -2$ (multiplicity 1), $r = \pm 2i$.

Consequently, $y_c = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(2t) + c_4\sin(2t)$ (c_1, c_2, c_3, c_4 arbitrary constants) is the general solution of $y^{(4)} - 16y = 0$.

The easiest way to get a particular solution of $y^{(4)} - 16y = 2t + e^t$ is to use the method of undetermined coefficients with a trial particular solution of the form $y_p(t) = y_{p_1}(t) + y_{p_2}(t) = At + B + Ce^t$ where $A, B,$ and C are constants to be determined so that $y_p^{(4)} - 16y_p = 2t + e^t$. (See the analysis above for y_{p_1} and y_{p_2}.) Differentiating four times, we have $y_p' = A + Ce^t$, $y_p'' = Ce^t$, $y_p''' = Ce^t$, $y_p^{(4)} = Ce^t$. Substituting in $y_p^{(4)} - 16y_p = 2t + e^t$ yields $Ce^t - 16(At + B + Ce^t) = 2t + e^t$. Rearranging, $-16At - 16B - 15Ce^t = 2t + 0 \cdot t^0 + 1 \cdot e^t$. Equating like coefficients gives $-16A = 2$, $-16B = 0$, and $-15C = 1$. Therefore $A = -\frac{1}{8}$, $B = 0$, and $C = -\frac{1}{15}$ so $y_p(t) = -\frac{1}{8}t - \frac{1}{15}e^t$. The general solution $y = y_c + y_p$ of the nonhomogeneous equation $y^{(4)} - 16y = 2t + e^t$ is therefore

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(2t) + c_4\sin(2t) - \frac{1}{8}t - \frac{1}{15}e^t}$$

where $c_1, c_2, c_3,$ and c_4 are arbitrary constants.

5.[14] One theory of epidemic spread postulates that the time rate of change in the infected population is proportional to the product of the number of individuals who have the disease with the number of disease free individuals. Assuming that the population of mice in a certain meadow has a stable value of one thousand, use this theory of epidemic spread to write – BUT NOT SOLVE - an initial value problem that models the number $N(t)$ of infected mice at time $t \geq 0$, if ten mice were initially infected.

Let $N(t)$ denote the number of infected mice in the meadow at time $t \geq 0$. Since the total population of mice in the meadow is constant at 1000, the number of disease free mice at time t is $1000 - N(t)$. The time rate of change in the infected population of mice is $\frac{dN}{dt}$. According to the first sentence in the description of this problem, we have

$$\frac{dN}{dt} = k\,N(t)\big(1000 - N(t)\big)$$

where k is a constant of proportionality. Since ten mice were initially infected, $N(0) = 10$. Thus, the I.V.P. modeling the epidemic is

$$\boxed{N' = kN(1000 - N), \quad N(0) = 10.}$$

6. [20] Determine the longest interval in which the initial value problem

$$ty''+\ln(t)y'=3+\frac{1}{(t-3)^2}, \quad y(2)=0,\ y'(2)=1$$

is guaranteed to have a unique, twice differentiable solution. Do not attempt to find the solution.

We will use the existence-uniqueness theorem for second-order linear I.V.Ps. Normalizing the DE yields

$$y''+\frac{\ln(t)}{t}y'=\frac{3}{t}+\frac{1}{t(t-3)^2}.$$

The coefficients and $g(t)$ for the normalized DE are continuous on the following intervals:

$p_0(t)=\frac{\ln(t)}{t}$ is continuous on $0<t<\infty$;

$p_1(t)=0$ is continuous on $-\infty<t<\infty$;

$g(t)=\frac{3}{t}+\frac{1}{t(t-3)^2}$ is continuous on $-\infty<t<0$, $0<t<3$, and $3<t<\infty$.

Therefore $0<t<3$ and $3<t<\infty$ are the common intervals of continuity for the coefficients and $g(t)$. But the initial conditions are given at the point $t_0=2$ and we must have t_0 in the interval. Thus

$$\boxed{0<t<3}$$

is the longest interval in which the given I.V.P. is guaranteed to have a unique, twice differentiable solution.

7. (a) [18] Solve the initial value problem $y''+y=\delta\left(t-\frac{\pi}{2}\right)-2\delta(t-\pi),\quad y(0)=1,\ y'(0)=0.$

(b) [3] Which is largest, $y(\pi/4)$, $y(3\pi/4)$, or $y(5\pi/4)$? Justify your answer.

(a) We use the method of Laplace transforms to solve the I.V.P. If $y=y(t)$ is a solution then by linearity and formulas 6 and 9 in the table of transforms,

$$\mathcal{L}\{y''+y\}(s)=\mathcal{L}\{\delta(t-\tfrac{\pi}{2})-2\delta(t-\pi)\}(s)$$

$$s^2\mathcal{L}\{y\}-sy(0)-y'(0)+\mathcal{L}\{y\}(s)=e^{-\frac{\pi}{2}s}-2e^{-\pi s}$$

Substituting the initial conditions and rearranging yields

$$(s^2+1)\mathcal{L}\{y\}(s)=s+e^{-\frac{\pi}{2}s}-2e^{-\pi s}$$

$$\mathcal{L}\{y\}(s)=\frac{s}{s^2+1}+e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+1}-2e^{-\pi s}\cdot\frac{1}{s^2+1}.$$

Taking the inverse Laplace transform and using formulas 4, 8, and 3 in the table of transforms yields the solution to the I.V.P.:

$$y(t)=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+1}\right\}-2\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=\cos(t)+u_{\frac{\pi}{2}}(t)\sin(t-\tfrac{\pi}{2})-2u_{\pi}(t)\sin(t-\pi)}.$$

(b) $$y(\tfrac{\pi}{4})=\cos(\tfrac{\pi}{4})+\overbrace{u_{\frac{\pi}{2}}(\tfrac{\pi}{4})}^{0}\sin(-\tfrac{\pi}{4})-2\overbrace{u_{\pi}(\tfrac{\pi}{4})}^{0}\sin(-\tfrac{3\pi}{4})=\boxed{\frac{\sqrt{2}}{2}}$$

$$y(\tfrac{3\pi}{4})=\cos(\tfrac{3\pi}{4})+\underbrace{u_{\frac{\pi}{2}}(\tfrac{3\pi}{4})}_{1}\sin(\tfrac{\pi}{4})-2\underbrace{u_{\pi}(\tfrac{3\pi}{4})}_{0}\sin(-\tfrac{\pi}{4})=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=0$$

$$y(\tfrac{5\pi}{4})=\cos(\tfrac{5\pi}{4})+\underbrace{u_{\frac{\pi}{2}}(\tfrac{5\pi}{4})}_{1}\sin(\tfrac{3\pi}{4})-2\underbrace{u_{\pi}(\tfrac{5\pi}{4})}_{1}\sin(\tfrac{\pi}{4})=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-2\frac{\sqrt{2}}{2}$$

$$=-\sqrt{2}$$

Therefore $\boxed{y(\tfrac{\pi}{4})\text{ is the greatest of the three.}}$

8.[20] Pure spring water flows into Pond 1 at a rate of 200 gallons per hour. Pond 1 initially contains 1,000,000 gallons of water and is contaminated by 100 grams of perchlorate. Solution from Pond 1 flows into Pond 2, which initially contains 5,000 gallons of water and 10 grams of perchlorate. The solution in Pond 2 then flows out into the river. The flow rates of the solutions exiting Ponds 1 and 2 are identical and are equal to 300 gallons per hour. Assuming that the distribution of perchlorate in each pond is uniform, write, BUT DO NOT SOLVE, a system of differential equations and initial conditions that models the amounts $Q_1(t)$ and $Q_2(t)$ of perchlorate in Ponds 1 and 2, respectively, at times t satisfying $0 \le t < 10{,}000$ hours.

We apply the principle Net Rate = Rate In − Rate Out to each of the ponds.

Pond 1: $$\frac{dQ_1}{dt} = \left(\frac{0 \text{ grams}}{\text{gal}}\right)\left(\frac{200 \text{ gal}}{\text{hr}}\right) - \left(\frac{Q_1(t) \text{ grams}}{(1{,}000{,}000 - 100t)\text{gal}}\right)\left(\frac{300 \text{ gal}}{\text{hr}}\right)$$

Pond 2: $$\frac{dQ_2}{dt} = \left(\frac{Q_1(t) \text{ grams}}{(1{,}000{,}000 - 100t)\text{gal}}\right)\left(\frac{300 \text{ gal}}{\text{hr}}\right) - \left(\frac{Q_2(t) \text{ grams}}{5000 \text{ gal}}\right)\left(\frac{300 \text{ gal}}{\text{hr}}\right)$$

Notes: It important to observe that the volume of solution in Pond 1 decreases by 100 gallons per hour so $V_1(t) = 1{,}000{,}000 - 100t$ is the volume of Pond 1 after t hours. On the other hand, the volume of solution in Pond 2 is constant over time, namely 5,000 gallons. Finally, the amounts of perchlorate initially present in Ponds 1 and 2 are, respectively, $Q_1(0) = 100$ grams and $Q_2(0) = 10$ grams.

Therefore, the system of DEs and I.C.s modeling the perchlorate amounts is

$$\boxed{\begin{aligned} Q_1' &= \frac{-3Q_1}{10{,}000 - t}, & Q_1(0) &= 100 \\ Q_2' &= \frac{3Q_1}{10{,}000 - t} - \frac{3Q_2}{50}, & Q_2(0) &= 10 \end{aligned}}$$

where t is in hours and Q_1 and Q_2 are in grams.

9.[21] Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$. Then assuming $\vec{x} = \vec{k}e^{rt}$ solves $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$.

The eigenvalues r of A satisfy

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-3)(r-1)+1 = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r = 2$ with multiplicity 2. An eigenvector $\vec{k}$ of A corresponding to the eigenvalue 2 satisfies $(A - 2I)\vec{k} = \vec{0}$ so $\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and hence

$$\begin{cases} \cancel{-k_1 - k_2 = 0} & \text{Redundant} \\ k_1 + k_2 = 0 \Rightarrow k_2 = -k_1. \end{cases}$$

Consequently, $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Taking $k_1 = 1$ we obtain $\vec{x}^{(1)} = \vec{k}e^{rt} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ as a solution to $\vec{x}' = A\vec{x}$. Up to a constant factor, this is the only solution of $\vec{x}' = A\vec{x}$ of the form $\vec{x} = \vec{k}e^{rt}$. To get a second linearly independent solution of $\vec{x}' = A\vec{x}$, we assume $\vec{x}^{(2)} = \vec{k}te^{rt} + \vec{\ell}e^{rt}$ where $\vec{k}$ and $\vec{\ell}$ are constant vectors and r is a scalar constant. Substituting this form for $\vec{x}^{(2)}$ into $\vec{x}' = A\vec{x}$ leads to the system $\begin{cases} (A - rI)\vec{k} = \vec{0}, \\ (A - rI)\vec{\ell} = \vec{k}. \end{cases}$ As above, $r = 2$ and $\vec{k} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ solve the first equation in the system. Substituting these into the second equation of the system gives $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or equivalently $\begin{cases} -\ell_1 - \ell_2 = 1 \\ \ell_1 + \ell_2 = -1. \end{cases}$

The first equation is redundant; the second gives $\ell_2 = -1 - \ell_1$ so

$$\vec{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} \ell_1 \\ -1-\ell_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \ell_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Without loss of generality, we may assume $\ell_1 = 0$ so $\vec{\ell} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Hence $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$

The general solution of $\vec{x}' = A\vec{x}$ is $\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{2t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{2t}\right)$ where c_1 and c_2 are arbitrary constants. Applying the initial conditions produces $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, from which it is clear that $c_1 = 1$ and hence $c_2 = -1$.

Thus $\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{2t} - \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{2t}\right) = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}te^{2t}}$ solves the I.V.P.

10. [21] Find the general solution of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then assuming $\vec{x} = \vec{k}e^{rt}$ solves the associated homogeneous equation $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$. The eigenvalues r of A satisfy

$$0 = \det(A - rI) = \begin{vmatrix} -r & 1 \\ -1 & -r \end{vmatrix} = r^2 + 1 \quad \Rightarrow \quad r = \pm i.$$

An eigenvector $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ of A corresponding to the eigenvalue i satisfies

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and thus } \begin{cases} -ik_1 + k_2 = 0 \quad \Rightarrow \quad k_2 = ik_1. \\ \cancel{-k_1 - ik_2 = 0} \leftarrow \text{Redundant; it's } -i \text{ times the first equation.} \end{cases}$$

Therefore $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ ik_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ i \end{bmatrix}$ and we may take $k_1 = 1$. Hence $\vec{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}e^{it} =$

$\begin{bmatrix} 1 \\ i \end{bmatrix}(\cos t + i\sin t) = \begin{bmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. Consequently $\vec{x}^{(1)}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\vec{x}^{(2)}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ are real solutions to $\vec{x}' = A\vec{x}$. Since $W(\vec{x}^{(1)}, \vec{x}^{(2)}) =$

$\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0$, they form a F.S.S. to $\vec{x}' = A\vec{x}$. That is,

$\vec{x}_c(t) = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ is the general solution of $\vec{x}' = A\vec{x}$.

To obtain a particular solution to $\vec{x}' = A\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it is easiest to use the method of undetermined coefficients. Since $\vec{g}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a solution to the homogeneous equation $\vec{x}' = A\vec{x}$, we may assume a trial particular solution of the nonhomogeneous equation of the form $\vec{x}_p = \vec{a}$ where $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is a constant vector to be determined. Then $\vec{x}_p' = \vec{0}$ so substituting in $\vec{x}_p' = A\vec{x}_p + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ leads to $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
This is equivalent to $\begin{cases} 0 = a_2 + 1 \\ 0 = -a_1 \end{cases}$ so $a_1 = 0$ and $a_2 = -1$. Thus $\vec{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The general solution of $\vec{x}' = A\vec{x} + \vec{g}(t)$ is $\vec{x} = \vec{x}_c(t) + \vec{x}_p(t) = \boxed{c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$
where c_1 and c_2 are arbitrary constants.

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}$, $n = 0,1,2,3\ldots.$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct}f(t)$	$F(s-c)$
8. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
9. $\delta(t-c)$	e^{-cs}

Math 204 Final Exam "Master List" Scorecard, 2013 Spring Semester

200 |
199
198
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196 |
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189 卌
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179 卌 |
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167 卌
166 卌 |||
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162 卌 |
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160 卌 ||
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157 ||
156 卌
155 |||
154 |||
153 卌 卌
152 |||
151 卌 |||
150 |

36 As (10.2 %)

85 Bs (24.0 %)

149 ||
148 卌 |
147 |||
146 |
145 ||
144 卌 |||
143 |||
142 ||
141 卌 |
140 卌 |||
139 ||||
138 |||
137 |||
136 ||
135 卌
134 卌
133 卌 ||
132 卌
131 ||
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129 ||||
128 |||
127 卌
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125 ||
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118 卌 |||
117 卌
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113 ||
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111 ||
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108 ||||
107 |
106 ||
105 ||
104 ||
103 |
102 |
101 |
100
99 |

81 Cs (22.9 %)

64 Ds (18.1 %)

98 ||
97
96 |
95
94 |
93 |
92
91
90
89 |||
88 |
87 |
86 ||||
85 |
84 |
83 |
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81 ||
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62 |
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54 |
53 ||
52
51
50
49
48

88 Fs (24.9 %)

47
46 ||
45 |
44
43
42
41
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31
30
29
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18
17
16
15
14
13
12
11
10
9
8 |
7 ||
6
5
4
3
2
1
0

Number taking final: 354
Median: 144
Mean: 139.8
Standard Deviation: 35.8

Math 204 Final Exam, 2013 Spring Semester, Instructor Grow, Section E

200		149		98	47
199		148 I		97	46
198		147		96	45
197		146		95	44
196 I		145		94	43
195	4 As	144	7 Cs	93	42
194		143		92	41
193		142		91	40
192	(10.5%)	141	(18.4%)	90	39
191		140 I		89	38
190		139 I		88	37
189		138 I		87	36
188		137 II		86	35
187		136		85	34
186		135		84	33
185		134 II		83	32
184		133 II		82	31
183 I		132 II		81	30
182 I		131		80 I	29
181		130		79	28
180 I		129 II		78	27
179 I		128 II		77	26
178 I		127		76	25
177		126		75	24
176		125	14 Ds	74	23
175	7 Bs	124		73	22
174		123	(36.8%)	72	21
173	(18.4%)	122		71	20
172		121		70	19
171 I		120		69	18
170		119 I		68	17
169		118		67	16
168		117 I		66	15
167		116 I		65	14
166 I		115		64	13
165 I		114		63	12
164 I		113		62	11
163 I		112		61	10
162		111 I		60	9
161		110		59 I	8
160		109		58	7
159		108	6 Fs	57	6
158		107		56	5
157		106	(15.8%)	55	4
156		105		54	3
155 I		104		53	2
154		103		52	1
153 I		102		51	0
152 I		101		50	
151 II		100		49	
150		99		48	

Number taking final: 38
Median: 138.5
Mean: 143.2
Standard Deviation: 27.8