Mathematics 204

Summer 2013

Final Exam

Your Printed Name: SOLUTION

Your Instructor's Name: ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 9 pages of problems containing 9 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown on integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [22] at the beginning of a problem indicates the point value of that problem is 22. The maximum possible score on this exam is 200.

problem	1	2	3	4	5	6	7	8	9	**Sum**
points earned										
maximum points	22	22	22	22	22	24	22	22	22	200

1.[22] Find the solution to $ty' = \frac{2}{t} - 2y$ for $t > 0$.

use integrating factor method:

$(t>0)$

$$ty' + 2y = \frac{2}{t} \Rightarrow y' + \frac{2}{t}y = \frac{2}{t^2}$$

$$\mu = e^{\int 2/t \, dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\Rightarrow \mu \cdot \left[y' + \frac{2}{t}y\right] = \mu \cdot \frac{2}{t^2} \Rightarrow (t^2 \cdot y)' = \cancel{t^2} \cdot \frac{2}{\cancel{t^2}} = 2$$

$$\int (t^2 y)' \, dt = \int 2 \, dt$$

$$t^2 y = 2t + C$$

$$y = \frac{2}{t} + \frac{C}{t^2}$$

2.[22] Find the explicit solution of the initial value problem $y' = \dfrac{2t^2 - t}{3y}$, $\quad y(1) = -\dfrac{5}{3}$.

use Variable-Separation's Method:

$y' = g(y)h(t)$ with $g(y) = \frac{1}{3y}$ $\quad h(t) = 2t^2 - t$

$\Rightarrow \int 3y\, dy = \int (2t^2 - t)\, dt$

$\frac{3}{2}y^2 = \frac{2}{3}t^3 - \frac{1}{2}t^2 + C$

$y^2 = \frac{4}{9}t^3 - \frac{1}{3}t^2 + \tilde{C}$ $\qquad (\tilde{C} = \frac{2}{3}C)$

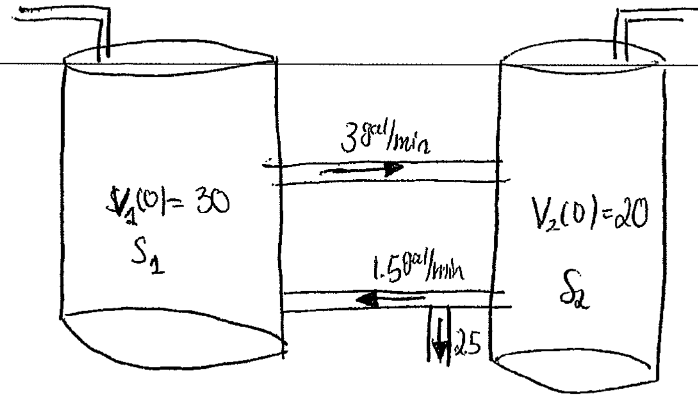
$y = \pm\sqrt{\frac{4}{9}t^3 - \frac{1}{3}t^2 + \tilde{C}}$

$-\frac{5}{3} = y(1) = -\sqrt{\frac{4}{9} - \frac{1}{3} + \tilde{C}} = -\sqrt{\frac{1}{9} + \tilde{C}} \Rightarrow \frac{25}{9} = \frac{1}{9} + \tilde{C} \Rightarrow \tilde{C} = \frac{24}{9} = \frac{8}{3}$

$\Rightarrow \quad y(t) = -\sqrt{\frac{4}{9}t^3 - \frac{1}{3}t^2 + \frac{8}{3}}$

3. [22] Consider the two interconnected tanks shown in the Figure below. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

Let $S_1(t)$ and $S_2(t)$, respectively, be the amount of salt in each tank at time t. Set up, **BUT DO NOT SOLVE**, an initial value problem that models the flow process.



$$S_1(0) = 25$$
$$S_2(0) = 15$$

$$S_1' = \frac{dS_1}{dt} = r_{salt}^{in} - r_{salt}^{out} = 1 \cdot 1.5 + 1.5 \cdot \frac{S_2(t)}{V_2(t)} - 3 \cdot \frac{S_1(t)}{V_1(t)}$$

$$S_2' = \frac{dS_2}{dt} = r_{salt}^{in} - r_{salt}^{out} = 3 \cdot 1 + 3 \cdot \frac{S_1(t)}{V_1(t)} - 4 \cdot \frac{S_2(t)}{V_2(t)}$$

$$V_1(t) = V_1(0) + t(1.5 + 1.5 - 3) = V_1(0) = 30$$
$$V_2(t) = V_2(0) + t(1 + 3 - 4) = V_2(0) = 20$$

$$S_1' = 1.5 + 1.5\,\frac{S_2(t)}{20} - 3 \cdot \frac{S_1(t)}{30} = \frac{3}{2} + \frac{3}{40} S_2(t) - \frac{S_1(t)}{10}$$

$$S_2' = 3 + 3\,\frac{S_1(t)}{30} - 4 \cdot \frac{S_2(t)}{20} = 3 + \frac{S_1(t)}{10} - \frac{1}{5} S_2(t)$$

4.[22] Solve $y'' - 2y' + y = \frac{e^t}{t^2}$ on the interval $t > 0$.

1.) $y'' - 2y' + y = 0$ gives homogeneous solution

$y(t) = e^{rt} \Rightarrow \underbrace{r^2 - 2r + 1}_{(r-1)^2} = 0 \Rightarrow r_1 = 1 = r_2$ (repeating roots)

$\Rightarrow y_1(t) = e^{1 \cdot t} = e^t \quad y_2(t) = t \cdot y_1(t) = te^t$

$\Rightarrow y_h = c_1 e^t + c_2 te^t$

2.) Find particular solution to $y'' - 2y' + y = \frac{e^t}{t^2}$ by using VoP. $y_p(t) = y_1 u_1(t) + y_2 u_2(t)$

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W(t)} dt \qquad u_2(t) = \int \frac{y_1(t) g(t)}{W(t)} dt$$

$$u_1(t) = -\int \frac{\cancel{te^t} e^t / t^2}{e^{2t}} dt = -\int \frac{1}{t} dt = -\ln t + \cancel{A}^{\,0}$$

$$\boxed{W(t) = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = \cancel{te^{2t}} + e^{2t} - \cancel{te^{2t}} = e^{2t}}$$

$$u_2(t) = \int \frac{\cancel{e^t} e^t / t^2}{\cancel{e^{2t}}} dt = \int \frac{1}{t^2} dt = -t^{-1} + \cancel{B}^{\,0}$$

$\Rightarrow y_p(t) = -(\ln t) e^t - (t^{-1}) te^t$ 3) $\Rightarrow y(t) = y_h + y_p = \overbrace{c_1 e^t + c_2 te^t + e^t(-\ln t) - e^t}^{\check{c} e^t}$

$= \check{c} e^t + c_2 te^t - (\ln t) e^t$

5.[22] Find the general solution of $y^{(4)} + y'' = \sin(2t)$.

1) $y^{(4)} + y'' = 0 \qquad y = e^{rt}$

$r^4 + r^2 = 0 \Rightarrow r^2(r^2+1) = 0 \qquad r_1 = r_2 = 0$ (repeat root)

$r_{3/4}^2 = -1 \Rightarrow r_3 = +i, \; r_4 = -i$

$\Rightarrow y_1(t) = e^{0t} = 1 \quad y_2(t) = te^{0t} = t \quad y_3 = \cos t \quad y_4 = \sin t$

$\Rightarrow y_h = C_1 + C_2 t + C_3 \cos t + C_4 \sin t$

2) use $\underline{MUC}$ given $y_p(t) = (A\cos(2t) + B\sin(2t)) \cdot t^s \quad s = 0$

$= A\cos(2t) + B\sin(2t)$

$\Rightarrow$ Note also $\quad y_p'' = -4A\cos(2t) - 4B\sin(2t)$

$y_p^{(4)} = 16A\cos(2t) + 16B\sin(2t)$

$\Rightarrow 16A\cos(2t) + 16B\sin(2t) - 4A\cos(2t) - 4B\sin(2t) = \sin(2t)$

$\Rightarrow \sin(2t)[16B - 4B] = 1 \cdot \sin(2t) \quad \Rightarrow 1 = \overbrace{16B - 4B}^{12B} \Rightarrow B = \frac{1}{12}$

$\cos(2t)[16A - 4A] = 0 \quad \Rightarrow A = 0$

$\Rightarrow y_p(t) = \frac{1}{12}\sin(2t)$

3) $y(t) = y_h + y_p = C_1 + C_2 t + C_3\cos t + C_4 \sin t + \frac{1}{12}\sin(2t)$

6. (a) [20] Solve the initial value problem $y'' + 4y = 2\delta(t-\pi) - \delta\left(t - \frac{\pi}{2}\right)$, $y(0) = 1$, $y'(0) = -1$ for $t > 0$.

(b) [4] Which is greater, $y(\pi/2)$ or $y(3\pi/2)$? Justify your answer.

a) $\mathcal{L}\{y''+4y\} = \mathcal{L}\{2\delta(t-\pi) - \delta(t-\pi/2)\} \Rightarrow \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-\pi)\} - \mathcal{L}\{\delta(t-\pi/2)\}$

$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=-1} + 4\mathcal{L}\{y\} = 2e^{-\pi s} - e^{-\pi/2\, s}$

$[s^2+4]\mathcal{L}\{y\} = s - 1 + 2e^{-\pi s} - e^{-\pi/2\, s}$

$\mathcal{L}\{y\} = \frac{s}{s^2+2^2} - \frac{1}{s^2+2^2} + e^{-\pi s}\frac{2}{s^2+2^2} - e^{-\pi/2\, s}\left(\frac{1}{s^2+2^2}\right)$

$y = \cos(2t) - \frac{1}{2}\sin(2t) + u_\pi(t)\sin(2(t-\pi)) - u_{\frac{\pi}{2}}(t)\frac{1}{2}\sin(2(t-\pi/2))$

Note $\sin(2(t-\pi)) = \sin(2t-2\pi) = \sin(2t)$ because sin is 2π-periodic

$y = \cos(2t) - \frac{1}{2}\sin(2t) + u_\pi(t)\sin(2t) - \frac{1}{2}u_{\frac{\pi}{2}}(t)\sin(2t-\pi)$

b) $y(\pi/2) = \cos(2(\frac{\pi}{2})) - \frac{1}{2}\sin(2(\pi/2)) + 0\cdot\sin(2(\frac{\pi}{2})) - \frac{1}{2}\cdot 1\sin(2(\frac{\pi}{2}) - \pi)$

$= \cos(\pi) - \underbrace{\frac{1}{2}\sin(\pi)}_{=0} - \underbrace{\frac{1}{2}\sin(0)}_{=0} = \cos(\pi) = \underline{\underline{-1}}$

$y(\frac{3}{2}\pi) = \cos(2(\frac{3}{2}\pi)) - \frac{1}{2}\sin(2(\frac{3}{2}\pi)) + 1\sin(2(\frac{3}{2}\pi)) - \frac{1}{2}\cdot 1\cdot\sin(2(\frac{3}{2}\pi) - \pi)$

$= \cos(3\pi) - \frac{1}{2}\cancel{\sin(3\pi)}^{0} + \cancel{\sin(3\pi)}^{0} - \frac{1}{2}\cancel{\sin(2\pi)}^{0} = \cos(3\pi) = \underline{\underline{-1}}$

$\Rightarrow y(\pi/2) = y(\frac{3}{2}\pi)$

7. [22] Find the solution of

$$y'+y=\begin{cases} 1 & \text{if } 0\le t<\pi, \\ 0 & \text{if } \pi\le t, \end{cases}$$

satisfying $y(0)=2$ for $y>0$.

Note $y'+y=g(t)$ $g(t)$ is off-function $\Rightarrow$ $g(t)=1-u_\pi(t)$

$\Rightarrow \mathcal{L}\{y'+y\}=\mathcal{L}\{1-u_\pi(t)\}$

$\mathcal{L}\{y'\}+\mathcal{L}\{y\}=\mathcal{L}\{1\}-\mathcal{L}\{u_\pi(t)\}$

$s\mathcal{L}\{y\}-\overset{2}{y(0)}+\mathcal{L}\{y\}=\frac{1}{s}-e^{-\pi s}\frac{1}{s}$

$\mathcal{L}\{y\}[s+1]=2+\frac{1}{s}-\frac{1}{s}e^{-\pi s}$

$\mathcal{L}\{y\}=\frac{2}{s+1}+\frac{1}{s(s+1)}-\frac{1}{s(s+1)}e^{-\pi s}$

P.F.D.

$\frac{1}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}=\frac{As+A+Bs}{s(s+1)}$

$\Rightarrow A=1 \Rightarrow B=-1$

$y(t)=2e^{-t}+\mathcal{L}^{-1}\{\frac{1}{s}-\frac{1}{s+1}\}-\mathcal{L}^{-1}\{(\frac{1}{s}-\frac{1}{s+1})e^{-\pi s}\}$

$y(t)=2e^{-t}+1-e^{-t}-u_\pi(t)\cdot[1-e^{-(t-\pi)}]$

8.[22] Solve system $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\mathbf{x}$.

1) $\det(A - \lambda I_2) = 0 = (3-\lambda)(-3-\lambda) - (-1)\cdot 9 = -\cancel{9} - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 + \cancel{9} =$

$= \lambda^2 \overset{!}{=} 0 \Rightarrow \lambda_1 = \lambda_2 = 0$

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2) $(A - 0\cdot I_2)\vec{v}^{(1)} \overset{!}{=} \vec{0} \Rightarrow \begin{array}{cc|c} 3 & 9 & 0 \\ -1 & -3 & 0 \end{array} \to \begin{array}{cc|c} 3 & 9 & 0 \\ 0 & 0 & 0 \end{array}$

$3v_1 + 9v_2 = 0 \to v_1 = -3v_2 \Rightarrow \begin{array}{l} v_2 = 1 \\ v_1 = -3 \end{array}$

$\Rightarrow \vec{v}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{0\cdot t} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$(A - 0\cdot I_2)\vec{w} \overset{!}{=} \vec{v}^{(1)} \Rightarrow \begin{array}{cc|c} 3 & 9 & -3 \\ -1 & -3 & 1 \end{array} \Rightarrow \begin{array}{cc|c} 3 & 9 & -3 \\ 0 & 0 & 0 \end{array}$

$\Rightarrow 3w_1 + 9w_2 = -3 \Rightarrow w_1 = -1 - 3w_2$

Pick $w_2 = 0 \Rightarrow w_1 = -1 \Rightarrow \vec{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{x}^{(2)} = t\cdot \vec{v}^{(1)} e^{0\cdot t} + \vec{w} e^{0\cdot t} = t \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

3)

$\Rightarrow \vec{x}(t) = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left[t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$

9.[22] Find the general solution of the system $\mathbf{x}' = \overbrace{\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}}^{=A}\mathbf{x} + \overbrace{\begin{pmatrix} -e^t \\ 2e^t \end{pmatrix}}^{b(t)}$, using the fundamental matrix $\mathbf{X} = \begin{pmatrix} e^{2t} & e^t \\ e^{2t} & 2e^t \end{pmatrix}$ for the associated homogeneous system.

$\vec{x}(t) = \mathbb{X}(t)\vec{c} + \vec{x}_p(t)$ Find $\vec{x}_p(t)$ using VoP-method.

$\vec{x}_p(t) = \mathbb{X}(t)\vec{u}(t)$ with $\vec{u}(t) = \int \mathbb{X}^{-1}(t) b(t)\,dt$

$$\Rightarrow \text{Need } \mathbb{X}^{-1}(t) = \frac{1}{\det \mathbb{X}(t)} \begin{bmatrix} 2e^t & -e^t \\ -e^{2t} & e^{2t} \end{bmatrix} = \frac{1}{(2e^{3t} - e^{3t})} \begin{bmatrix} 2e^t & -e^t \\ -e^{2t} & e^{2t} \end{bmatrix} =$$

$$= \frac{1}{e^{3t}} \begin{bmatrix} 2e^t & -e^t \\ -e^{2t} & e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} & -e^{-2t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

$$\Rightarrow \vec{u}(t) = \int \begin{bmatrix} 2e^{-2t} & -e^{-2t} \\ -e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} -e^t \\ 2e^t \end{bmatrix} dt =$$

$$= \int \begin{pmatrix} -2e^{-t} - 2e^{-t} \\ e^0 + 2e^0 \end{pmatrix} dt = \int \begin{pmatrix} -4e^{-t} \\ 3 \end{pmatrix} dt = \begin{pmatrix} 4e^{-t} \\ 3t \end{pmatrix}$$

$$\Rightarrow \vec{x}_p(t) = \mathbb{X}(t)\vec{u}(t) = \begin{bmatrix} e^{2t} & e^t \\ e^{2t} & 2e^t \end{bmatrix} \begin{pmatrix} 4e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} 4e^t + 3te^t \\ 4e^t + 6te^t \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = \mathbb{X}\vec{c} + \vec{x}_p(t) = c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} + \begin{pmatrix} 4e^t + 3te^t \\ 4e^t + 6te^t \end{pmatrix} =$$

$$= c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^t \begin{pmatrix} 4 \\ 4 \end{pmatrix} + te^t \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}$, $n = 0,1,2,3\ldots.$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $(f*g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}

Math 204 Final Exam "Master List" Scorecard, 2013 Summer Semester

200	1		149	1		98		47	
199			148	1		97		46	
198			147			96		45	
197			146	1		95		44	
196	2		145		8 Cs	94	1	43	
195	1		144			93		42	
194	1		143		(15.1 %)	92		41	
193	1		142	1		91		40	
192	1		141			90	1	39	
191		12 As	140	1		89		38	
190	1		139	1		88	1	37	
189		(22.6 %)	138	1		87		36	
188	2		137			86		35	
187			136			85		34	
186			135			84		33	
185	1		134			83		32	
184			133			82		31	
183			132	2		81		30	
182	1		131			80		29	
181			130	1		79		28	
180			129		10 Ds	78		27	
179	1		128	1		77		26	
178			127	1	(18.9 %)	76		25	
177	1		126			75		24	
176	1		125			74	1	23	
175	1		124			73		22	
174	1		123			72		21	
173	1		122			71		20	
172	2		121	1		70		19	
171		15 Bs	120	2		69		18	
170	1		119			68		17	
169			118			67		16	
168		(28.3 %)	117			66		15	
167			116			65		14	
166	1		115			64		13	
165	1		114			63		12	
164	1		113			62		11	
163			112	1		61		10	
162	1		111	1		60		9	
161	2		110			59		8	1
160			109		8 Fs	58		7	
159			108	1		57		6	
158	1		107		(15.1 %)	56		5	
157			106			55		4	
156	1		105			54		3	
155	1		104			53		2	
154			103			52		1	
153			102			51		0	
152			101			50			
151			100			49			
150			99			48			

Number taking final:_53___
Median: ___161_
Mean: _151.16____
Standard Deviation: _37.69_