Sec. 2.5 Autonomous Equations and Population Dynamics

HW p. 88: # 3, 9, 22, 28 Due: Monday, Sept. 13

Schaum's: (pp. 11-12, #2.10; p.57 #7.7; p.69, #7.35; p.72, #7.87, 7.88)

1^{st} order DEs: $y' = f(t,y)$

1^{st} order linear DEs: $y' = -p(t)y + q(t)$

1^{st} order separable DEs: $y' = g(t)h(y)$

$\boxed{1^{st} \text{ order } \underline{\text{automous}} \text{ DEs: } \quad y' = h(y)}$

Notes: ① The RHS does not depend on the independent variable t. Hence the DE has the name "autonomous" [aut (Gk. "self") + nomos (Gk. "law")] self-governing or independent

② Autonomous DEs are a special case of separable DEs. Therefore

$$\int \frac{dy}{h(y)} = \int dt$$

is a solution (up to quadrature).

Examples of Autonomous DE's:

Models for population growth

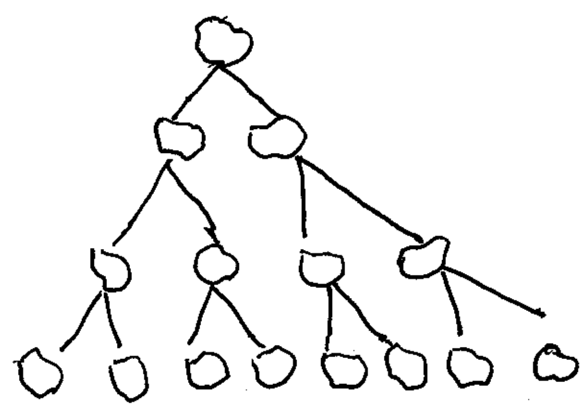
① $\dfrac{dy}{dt} = ky$ (law of natural growth/decay) $(k>0)$ $(k<0)$

② $\dfrac{dy}{dt} = (a-by)y$ (logistic equation)

③ $\dfrac{dy}{dt} = y^2(4-y^2)$ (#12, p. 89)

Ex 1| Consider the growth of an amoeba population that doubles every hour.

t (hrs.)	$P(t)$
0	1
1	2
2	4
3	8
⋮	⋮
t	2^t



$$y = P(t) = 2^t$$

$$\frac{dy}{dt} = \ln(2)e^{t\ln(2)} = \ln(2)2^t = \ln(2)y$$

$$\boxed{\frac{dy}{dt} = ky} \qquad \text{where} \quad k = \ln(2) > 0.$$

This is typical of population growth under ideal circumstances: no predators, plentiful food, unlimited space, etc. The general solution is

$$y(t) = y_0 e^{kt}$$

where $y_0 = y(0)$ is the initial population. To account for circumstances that limit the growth of populations, more complicated mathematical models were developed. A fairly simple successful model is the logistic equation:

Best to use numbers, say $\boxed{\frac{dy}{dt} = (2-3y)y}$

$$(*) \qquad \frac{dy}{dt} = (\overbrace{a-by}^{k(y)})y \qquad \text{(where } a \text{ and } b \text{ are positive constants)}$$

Ex 2 | (d) Find the general solution of the logistic equation (*).

(a) Find the equilibrium solutions of " " " (*).

(e) Sketch typical solution curves for " " " (*).

(b) Sketch a phase line for the logistic equation (*).

(c) Classify each of the equilibrium solutions as asymptotically stable, asymptotically unstable, or asymptotically semistable.

$(*)\qquad y' = (3-2y)y$

(a) Equilibrium solutions of $(*)$ are constant functions. Hence $y'=0$ so

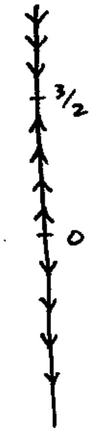
$$0 = (3-2y)y \implies \boxed{y=0 \text{ or } y = 3/2.}$$

(b)

Interval	Sign of $y' (= (3-2y)y)$
$-\infty < y < 0$	$-$
$0 < y < 3/2$	$+$
$3/2 < y < \infty$	$-$

Phase Line of $(*)$

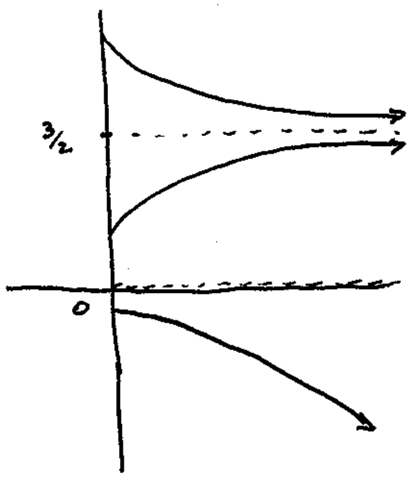
(c)



$3/2 \leftarrow$ Stable

$0 \leftarrow$ Unstable

(e)



Typical solution curves of $(*)$

(d) To find the general solution of $(*)$ we separate variables:

$$\int \frac{dy}{(3-2y)y} = \int dt$$

partial fraction decomposition

$$\frac{1}{(3-2y)y} = \frac{A}{3-2y} + \frac{B}{y} \implies 1 = Ay + B(3-2y)$$

To find B, set $y=0$: $\quad 1 = 0 + B(3) \implies B = 1/3$

To find A, set $y = \frac{3}{2}$: $\quad 1 = A\left(\frac{3}{2}\right) + 0 \implies A = 2/3$

$$\int \left(\frac{2/3}{3-2y} + \frac{1/3}{y} \right) dy = \int dt$$

$$-\frac{1}{3}\ln|3-2y| + \frac{1}{3}\ln|y| = t + \tilde{C}$$

(Defines solution $y=y(t)$ of $(*)$ implicitly.)

$$\ln\left|\frac{y}{3-2y}\right| = 3t + c$$

$$\frac{y}{3-2y} = Ae^{3t} \qquad (A = e^c)$$

$$y = Ae^{3t}(3-2y)$$

$$y = 3Ae^{3t} - 2Ae^{3t}y$$

$$y + 2Ae^{3t}y = 3Ae^{3t}$$

$$y(1+2Ae^{3t}) = 3Ae^{3t}$$

$$y(t) = \frac{3Ae^{3t}}{1+2Ae^{3t}}$$

$$\boxed{y(t) = \frac{3e^{3t}}{B+2e^{3t}}} \qquad \left(B = \tfrac{1}{A} \text{ is an arbitrary constant}\right)$$

(Explicit general solution of (*))

Ex 3 (#12, p. 89) For the autonomous DE

$$\frac{dy}{dt} = y^2(4-y^2),$$

(a) determine the equilibrium solutions;

(b) draw the phase line;

(c) classify the equilibrium solutions as asymptotically stable, unstable, or semistable;

(d) sketch several typical graphs of solutions of the DE in the ty-plane.

(a) $0 = y^2(4-y^2) = y^2(2-y)(2+y)$

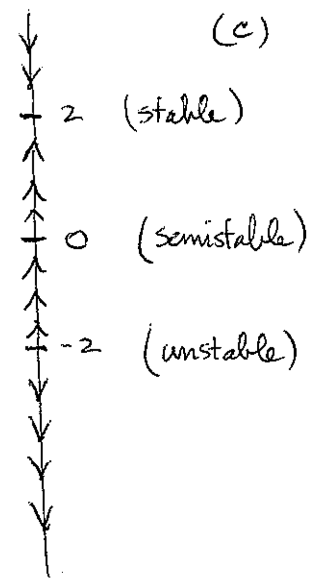
$$\boxed{y=0,\ y=2,\ \text{and}\ y=-2}$$

(b)

intervals of y-values	sign of $y' = y^2(4-y^2)$
$2 < y < \infty$	$-$
$0 < y < 2$	$+$
$-2 < y < 0$	$+$
$-\infty < y < -2$	$-$

Phase line

(c)



(d)