Mathematics 3304

Fall 2014

Exam I

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 6 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

problem	0	1	2	3	4	5	6	7	**Sum**
points earned									
maximum points	2	14	14	14	14	14	14	14	100

1. [14] Classify each differential equation by completing the columns in the following table. For each nonlinear equation, **circle** the term(s) which makes it nonlinear.

Differential Equation	Order?	Linear? (Y/N)	Homogeneous? (Y/N)
$\frac{d^2y}{dt^2}+\sin(t+y)=e^t$	2	N	
$t^2y^{(5)}+\cos(t)y-t^3=0$	5	Y	N
$\frac{d^3y}{dt^3}+\frac{1}{y}=0$	3	N	
$e^u u'''+e^{-t}u=1+t^2$	3	N	
5 pts.	4 pts.	4 pts.	1 pt.

(Please **DO NOT SOLVE** any of these equations.)

2. [14] Consider the differential equation $y' = y(4-2^y)(y+3)$.

(a) Find the equilibrium (or critical) points.

(b) Sketch the phase line (or phase portrait). **SHOW YOUR WORK.**

(c) Classify each equilibrium point as asymptotically stable, unstable, or semi-stable. **SHOW YOUR WORK.**

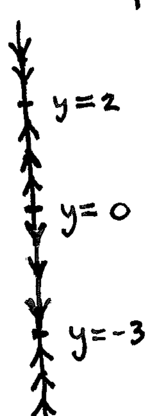
(d) If $y(0) = -1$, what is the limit of the solution $y(t)$ as t goes to infinity? **EXPLAIN WHY.**

(a) The equilibrium points are the constant solutions of the DE. For those solutions $y = y(t) =$ constant, we have $y' = 0$. Therefore they must satisfy

$$0 = y(4-2^y)(y+3),$$

So either $y = 0$ or $4 - 2^y = 0$ or $y + 3 = 0$. Hence $\boxed{y=0,\ y=2,\ y=-3}$ are the equilibrium points.

(b)



Phase Line

Interval	sign of y' $(= y(4-2^y)(y+3))$
$2 < y < \infty$	$(+)(-)(+) = -$
$0 < y < 2$	$(+)(+)(+) = +$
$-3 < y < 0$	$(-)(+)(+) = -$
$-\infty < y < -3$	$(-)(+)(-) = +$

(c) $\boxed{y = 2 \text{ is asymptotically stable}}$ because nearby solutions approach 2.

$\boxed{y = 0 \text{ is unstable}}$ because nearby solutions move away from 0.

$\boxed{y = -3 \text{ is asymptotically stable}}$ because nearby solutions approach -3.

(d) $\boxed{\lim_{t\to\infty} y(t) = -3}$ because a solution starting at -1 initially would approach the asymptotically stable equilibrium point $y = -3$.

3. [14] Solve the initial value problem $ty' = y + t^3 \sin(t)$, $y(\pi) = 0$.

The DE is a first order, linear equation. Rewrite it as

$$ty' - y = t^3 \sin(t)$$

1 pt. to here

$$y' - \frac{1}{t}y = t^2 \sin(t).$$ (Note: The IVP will have a unique soln. on $t>0$.)

4 pts. to here. An integrating factor is $e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln(t) + \cancel{c}^{\,0}} = e^{\ln(t^{-1})} = t^{-1}$.

Multiplying through by the integrating factor yields

$$t^{-1}\left(y' - t^{-1}y\right) = t^{-1}\left(t^2 \sin(t)\right)$$

5 pts. to here.

$$(*) \qquad t^{-1}y' - t^{-2}y = t\sin(t)$$

Note that the left member is exact according to the product rule:

$$(t^{-1}y)' = t^{-1}y' + (-1)t^{-2}y.$$

Substituting this for the left member in $(*)$ gives

$$(t^{-1}y)' = t\sin(t).$$

Integrating both sides with respect to t leads to

11 pts. to here. (2+4)

$$t^{-1}y = \int (t^{-1}y)'dt = \int \underbrace{t}_{v}\,\underbrace{\sin(t)dt}_{dv} = -t\cos(t) - \int -\cos(t)dt$$

$$= -t\cos(t) + \sin(t) + c$$

Multiplying through by t, we have

12 pts. to here

$$y(t) = ct + t\sin(t) - t^2\cos(t).$$

Applying the initial condition,

$$0 = y(\pi) = c\pi + \pi\sin(\pi) - \pi^2\cos(\pi) = c\pi + \pi^2.$$

13 pts. to here.

Therefore $c = -\pi$, so the solution of the IVP is

14 pts. to here.

$$\boxed{y(t) = t\sin(t) - \pi t - t^2\cos(t)}.$$

4. [14] Find the explicit solution of the differential equation $y' = \sqrt{1-y}$.

1 pts. to here

The DE is a first order, separable equation. (In fact, it is autonomous: $y' = f(y)$.) Writing

$$\frac{dy}{dt} = \sqrt{1-y}$$

and separating variables gives

$$\frac{dy}{\sqrt{1-y}} = dt. \qquad \text{(Note: This step is invalid if } y=1\text{. See below.)}$$

5 pts. to here.

Integrating both sides yields

$$-\frac{(1-y)^{1/2}}{1/2} + c_1 = -\int (1-y)^{-1/2} d(1-y) = \int dt = t + c_2.$$

(6 + 8)

12 pts. to here.

Simplifying leads to

$$-2\sqrt{1-y} = t + c_2 - c_1$$

$$\sqrt{1-y} = -\frac{1}{2}t + c \qquad \left(c = \frac{c_2 - c_1}{-2}\right)$$

$$1 - y = \left(c - \frac{t}{2}\right)^2$$

$$\boxed{y(t) = 1 - \left(c - \frac{t}{2}\right)^2} \quad \text{where } c \text{ is an arbitrary constant.}$$

14 pts. to here.

<u>Note 1</u>: This can be written in the equivalent form

$$y(t) = 1 - c^2 + ct - \frac{t^2}{4}$$

where c is an arbitrary constant.

<u>Note 2</u>: By inspection, it is easy to see that the constant function $\boxed{y(t) = 1}$ is also a solution to $y' = \sqrt{1-y}$. The solution $y(t) = 1$ is called a singular solution of $y' = \sqrt{1-y}$ because it cannot be obtained from the general solution $y(t) = 1 - \left(c - \frac{t}{2}\right)^2$ by any choice of the abitrary constant c.

5. [14] A 500 gallon tank originally contains 200 gallons of pure water. Then water containing two pounds of salt per gallon is poured into the tank at a rate of four gallons per minute, and the well-stirred mixture leaves at a rate of five gallons per minute. Write, **BUT DO NOT SOLVE**, an initial value problem for the amount $Q(t)$ of salt in the tank at time t.

Net Rate = Rate In − Rate Out

$$\frac{dQ}{dt} = \left(\frac{2\text{ lbs.}}{\text{gal}}\right)\left(\frac{4\text{ gal}}{\text{min.}}\right) - \left(\frac{5\text{ gal.}}{\text{min}}\right)\left(\frac{Q(t)\text{ lbs.}}{V(t)\text{ gal.}}\right)$$

The volume of solution in the tank at time t is $V(t) = 200 - t$ since the volume starts out at 200 gal. and decreases by 1 gal./min. thereafter. Consequently, an IVP that models the number $Q(t)$ of pounds of salt in the tank after t minutes is

$$\boxed{Q' = 8 - \left(\frac{5}{200-t}\right)Q \ , \quad Q(0) = 0} \ .$$

6. [14] Find the general solution of each equation and describe the behavior of the solution as $t \to \infty$.

1 pt. to here

(a) $y'' + 4y' + 5y = 0$ $\quad y = e^{rt}$ leads to $r^2 + 4r + 5 = 0$. By the quadratic formula,

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$ 2 pts. to here. Therefore the general solution is

6 pts. to here.

$$\boxed{y(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)} \quad (c_1, c_2 \text{ arbitrary constants}).$$

Clearly $e^{-2t} \to 0$ and $\cos(t)$ and $\sin(t)$ are bounded as $t \to \infty$. The squeeze theorem then implies $\boxed{y(t) \to 0 \text{ as } t \to \infty}$.

7 pts. to here.

1 pt. to here

(b) $y'' + 2y' + y = 0$ $\quad y = e^{rt}$ leads to $r^2 + 2r + 1 = 0$ or $(r+1)(r+1) = 0$.

2 pts. to here

Therefore $r = -1$ with multiplicity two. Thus the general solution is

6 pts. to here

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t}} \quad (c_1, c_2 \text{ arbitrary constants}).$$

Clearly $e^{-t} \to 0$ as $t \to \infty$. L'Hospital's rule shows that $te^{-t} \to 0$ as $t \to \infty$.

Therefore $\boxed{y(t) \to 0 \text{ as } t \to \infty}$.

7 pts. to here.

7. [14] Given that $y_1(t) = t$ is a solution of the differential equation

$$(*) \qquad t^2 y'' - t(t+2)y' + (t+2)y = 0$$

on the interval $t > 0$, use reduction of order to find a second solution y_2. Then verify that y_1 and y_2 form a fundamental set of solutions for the differential equation on the interval $t > 0$.

Assume $y_2(t) = u(t)y_1(t) = tu(t)$ where u is a nonconstant function to be determined so that y_2 solves $(*)$. Then

$$y_2' = tu' + 1u$$
$$y_2'' = tu'' + 1\cdot u' + u' = tu'' + 2u'$$

We want

$$0 = t^2 y_2'' - t(t+2)y_2' + (t+2)y_2,$$

so substituting the expressions for y_2 and its derivatives gives

$$0 = t^2(tu'' + 2u') - t(t+2)(tu' + u) + (t+2)tu$$

$$0 = t^3u'' + (2t^2 - t(t+2)t)u' + (-t(t+2) + (t+2)t)u$$

$$0 = t^3u'' - t^3u'.$$

Dividing through by t^3 yields a constant coefficient, homogeneous, linear DE:

$$0 = u'' - u'.$$

Then $u(t) = e^{rt}$ leads to $0 = r^2 - r = r(r-1)$ so $r = 0$ or $r = 1$.

Therefore $u(t) = c_1 + c_2 e^t$. Hence $y_2(t) = u(t)y_1(t) = (c_1 + c_2e^t)t = c_1t + c_2te^t$.

In order to obtain a second solution "different" from $y_1(t) = t$ we choose $c_1 = 0$ and $c_2 = 1$. I.e. $\boxed{y_2(t) = te^t}$.

In order to verify that $y_1(t) = t$ and $y_2(t) = te^t$ form a F.S.S. to $(*)$ on the interval, we compute their Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix} = t(t+1)e^t - te^t = \boxed{t^2e^t > 0} \quad \text{on } t > 0.$$

Therefore y_1 and y_2 form a F.S.S. of $(*)$ on $t > 0$.

2014 Fall Semester, Math 3304 Hour Exam I, Master List

100	III
99	IIII
98	III
97	卌 卌 卌 I
96	卌 IIII
95	卌 卌 卌 I
94	卌 卌 卌 II
93	卌 IIII
92	卌 卌 卌 III
91	卌 卌 IIII
90	卌 卌 卌 III
89	卌 卌 I
88	卌 卌 II
87	卌 卌 IIII
86	卌 卌 卌 卌 I
85	卌 卌 IIII
84	卌 卌 III
83	卌 卌 卌 II
82	卌 卌
81	卌 卌
80	卌 卌
79	卌 I
78	卌 卌 II
77	卌 卌 II
76	卌 III
75	卌 I
74	卌 卌 I
73	卌 IIII
72	卌 卌 I
71	卌 II
70	III
69	卌
68	卌 IIII
67	卌 II
66	卌
65	卌 II
64	II
63	卌 I
62	IIII
61	卌 III
60	卌
59	II
58	II
57	III
56	III
55	III
54	II
53	卌
52	I
51	I
50	II
49	IIII
48	I
47	I
46	I
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Number taking exam: 443
Median: 83
Mean: 79.3
Standard Deviation: 14.6

Number receiving A's: 127 — 28.7%
Number receiving B's: 132 — 29.8
Number receiving C's: 85 — 19.2
Number receiving D's: 58 — 13.1
Number receiving F's: 41 — 9.3

2014 Fall Semester, Math 3304 Hour Exam I
Instructor Grow, Section J

100		
99		
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97		
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95	I	8 As
94	II	
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92	I	
91	I	
90	III	
89	I	
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87	II	
86	I	
85	I	11 Bs
84		
83		
82	I	
81	II	
80	I	
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78	I	
77	I	
76	I	
75	II	9 Cs
74	I	
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72	I	
71	II	
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Number taking exam: 36
Median: 80.5
Mean: 77.6
Standard Deviation: 13.3

Number receiving A's: 8 — 22.2%
Number receiving B's: 11 — 30.6
Number receiving C's: 9 — 25.0
Number receiving D's: 4 — 11.1
Number receiving F's: 4 — 11.1