Math 3304 Fall 2015 Exam 2

Your printed name: Dr. Grow

Your instructor's name: ______

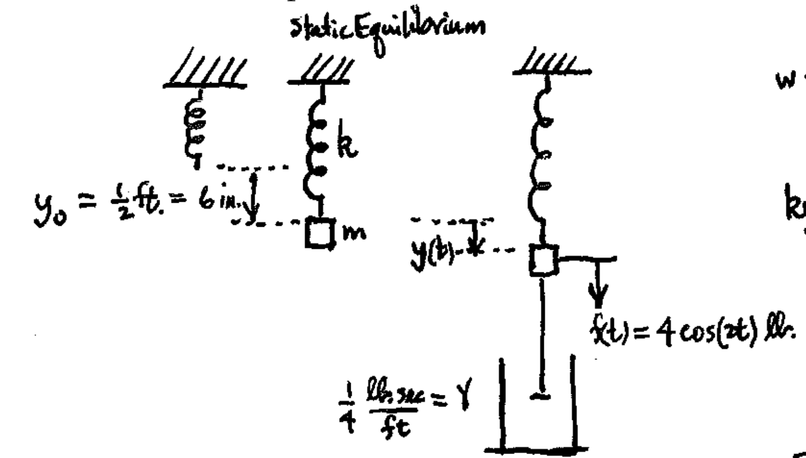
Your section (or Class Meeting Days and Time): ______

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noise making devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 2 consists of this cover page, 4 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] (Please use 32 ft/sec^2 as the acceleration of gravity in this problem) A spring, hanging vertically from a rigid support, is stretched 6 *in* by a mass that weighs 8 *lb*. The mass is in a medium with a damping constant of 0.25 *lb.s/ft* and is acted on by an external downward force of $4\cos 2t$ lb. Suppose that the mass is displaced an additional 2 ft in the upward direction and then released. Then set up , **BUT DO NOT SOLVE**, an initial value problem that models the motion of the body.



$$w = mg \Rightarrow m = \frac{w}{g} = \frac{8\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{4}\text{ slug.}$$

$$ky_0 = w \Rightarrow k = \frac{w}{y_0} = \frac{8\text{ lb.}}{\frac{1}{2}\text{ft.}} = 16\,\frac{\text{lb}}{\text{ft}}$$

$$my'' + \gamma y' + ky = f(t)$$

$$\boxed{\frac{1}{4}y'' + \frac{1}{4}y' + 16y = 4\cos(2t)\ ,\ y(0) = -2,\ y'(0) = 0}$$

y(t) = displacement of body from static equilibration position at time $t>0$

(y in feet, t is seconds)

2. [20] Find the inverse Laplace transform of $\dfrac{5s+1}{s^2-2s+3} = F(s)$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5s+1}{(s-1)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{5(s-1)+6}{(s-1)^2+2}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2}\right\} + \frac{6}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-1)^2+2}\right\}$$

$$\therefore \boxed{\mathcal{L}^{-1}\{F(s)\} = 5e^{t}\cos(\sqrt{2}t) + 3\sqrt{2}e^{t}\sin(\sqrt{2}t)}$$

(Formulas 9 and 8 in the table of Laplace transforms)

3. [20] Solve the initial value problem

$$t^2y'' - 3ty' + 4y = 0, \quad y(1) = 4, \quad y'(1) = 5.$$

$y = t^m$ in $t^2y'' - 3ty' + 4y = 0$ leads to $m(m-1) - 3m + 4 = 0$.

Therefore $0 = m^2 - 4m + 4 = (m-2)^2$ so $m = 2$ with multiplicity 2. The general solution to the DE is $y(t) = c_1 t^2 + c_2 t^2 \ln(t)$ for $t > 0$.

Then $y'(t) = 2c_1 t + 2c_2 t \ln(t) + c_2 t^2 \cdot \frac{1}{t} = (2c_1 + c_2)t + 2c_2 t \ln(t)$.

$4 = y(1) = c_1(1)^2 + c_2(1)^2 \ln(1) = c_1$

$5 = y'(1) = (2c_1 + c_2)(1) + 2c_2(1)\ln(1) = 2c_1 + c_2 = 8 + c_2 \Rightarrow -3 = c_2$.

Thus $\boxed{y(t) = 4t^2 - 3t^2 \ln(t)}$ solves the IVP.

4. [20] Find the general solution of

$$y^{(4)} + 3y'' - 4y = 2t + e^{2t}.$$

$y = e^{rt}$ in $y^{(4)} + 3y'' - 4y = 0$ leads to $r^4 + 3r^2 - 4 = 0$ so $(r^2+4)(r^2-1) = 0$

$\Rightarrow r = 1, -1, 2i, -2i$. Therefore $y_h(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(2t) + c_4 \sin(2t)$.

Since $g(t) = 2t + e^{2t}$, a trial particular solution is $y_p = At + B + Ce^{2t}$ where

A, B, and C are coefficients to be determined. Then

$$y_p' = A + 2Ce^{2t}, \quad y_p'' = 4Ce^{2t}, \quad y_p''' = 8Ce^{2t}, \quad y^{(4)} = 16Ce^{2t}.$$

We want $y_p^{(4)} + 3y_p'' - 4y_p = 2t + e^{2t}$ so

$$16Ce^{2t} + 3(4Ce^{2t}) - 4(At + B + Ce^{2t}) = 2t + e^{2t}$$

$$-4At - 4B + 24Ce^{2t} = 2t + e^{2t}.$$

Therefore $-4A = 2$, $-4B = 0$, and $24C = 1$. That is, $A = -\frac{1}{2}$, $B = 0$,

and $C = \frac{1}{24}$, so $y_p(t) = -\frac{t}{2} + \frac{1}{24}e^{2t}$. Hence, the general solution is

$$y(t) = y_h(t) + y_p(t)$$

$$\Rightarrow \boxed{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(2t) + c_4 \sin(2t) - \frac{t}{2} + \frac{1}{24}e^{2t}}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

5. [20] Use the Laplace transform to solve the initial value problem

$$y'' + 4y = 5\sinh t, \quad y(0) = y'(0) = 0.$$

$$\mathcal{L}\{y'' + 4y\}(s) = \mathcal{L}\{5\sinh(\cdot)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - \cancel{s y(0)}^{0} - \cancel{y'(0)}^{0} + 4\mathcal{L}\{y\}(s) = \frac{5}{s^2 - 1^2}$$

$$(s^2+4)\mathcal{L}\{y\}(s) = \frac{5}{s^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s^2-1)(s^2+4)}\right\}$$

$$\frac{5}{(s^2-1)(s^2+4)} = \frac{5}{(s-1)(s+1)(s^2+4)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4}$$

$$5 = A(s+1)(s^2+4) + B(s-1)(s^2+4) + (Cs+D)(s^2-1)$$

$s=1 \Rightarrow \quad 5 = A(2)(5) \Rightarrow A = 1/2.$

$s=-1 \Rightarrow \quad 5 = B(-2)(5) \Rightarrow B = -1/2.$

$s=2i \Rightarrow \quad 5+0i = (2iC+D)(-5) \Rightarrow -5D = 5$ and $-10C = 0$

so $D=-1$ and $C=0$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2+4}\right\}$$

$$\boxed{y(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - \frac{1}{2}\sin(2t)}$$

(Formulas 2 and 4 in the table of Laplace transforms)

or equivalently,

$$\boxed{y(t) = \sinh(t) - \frac{1}{2}\sin(2t)}$$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
10.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
11.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
12.	$u_c(t)\,f(t-c)$	$e^{-cs}F(s)$
13.	$e^{ct}f(t)$	$F(s-c)$

Math 3304

Exam II

Fall 2015

number of exams = 29

mean = 77.2

median = 80

Standard deviation = 17.0

Distribution of Scores:

Range	Grade	Frequency
90-100	A	8
80-89	B	8
70-79	C	5
60-69	D	3
0-59	F	5