Mathematics 3304

Fall 2014

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page and 6 pages of problems containing 6 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

problem	1	2	3	4	5	6	**Sum**
points earned							
maximum points	17	17	17	17	16	16	100

1. [17] Find the general solution of $y'' - 6y' + 9y = t^{-3}e^{3t}$ on the interval $t > 0$.

1 pt. to here $y = e^{rt}$ in $y'' - 6y' + 9y = 0$ leads to $r^2 - 6r + 9 = 0$. Therefore

2 pts. to here. $(r-3)^2 = 0$ so $r = 3$ with multiplicity two. Consequently

4 pts. to here. $$y_h(t) = c_1 e^{3t} + c_2 t e^{3t} \qquad (c_1, c_2 \text{ arbitrary constants})$$

is the general solution of the homogeneous equation $y'' - 6y' + 9y = 0$. We use variation of parameters to find a particular solution of the nonhomogeneous equation:

5 pts. to here.
$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $\{y_1(t), y_2(t)\} = \{e^{3t}, te^{3t}\}$ is a fundamental set of solutions to the associated homogeneous equation,

7 pts. to here.
$$u_1(t) = \int \frac{-y_2(t)g(t)}{W(t)}\,dt \quad \text{and} \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}\,dt.$$

Here $g(t) = t^{-3}e^{3t}$ and

9 pts. to here $$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{vmatrix} = (3t+1)e^{6t} - 3te^{6t} = e^{6t}.$$

Therefore

12 pts. to here. $$u_1(t) = \int \frac{-te^{3t}t^{-3}e^{3t}}{e^{6t}}\,dt = \int -t^{-2}\,dt = \frac{1}{t} + \cancel{c_1}^{\,0}$$

and

15 pts. to here. $$u_2(t) = \int \frac{e^{3t}t^{-3}e^{3t}}{e^{6t}}\,dt = \int t^{-3}\,dt = -\frac{1}{2t^2} + \cancel{c_2}.$$

16 pts. to here Thus $y_p(t) = t^{-1}e^{3t} - \frac{1}{2}t^{-2}\cdot te^{3t} = \frac{1}{2}t^{-1}e^{3t}$. The general solution of the nonhomogeneous equation is $y = y_h + y_p$ or

17 pts. to here. $$\boxed{y(t) = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{2}t^{-1}e^{3t}} \qquad (c_1, c_2 \text{ arbitrary constants}).$$

2.[17] Solve $y''-2y'+2y=2\cos(2t)$.

1 pt. to here. $y=e^{rt}$ in $y''-2y'+2y=0$ leads to $r^2-2r+2=0$. Therefore

3 pts. to here. $$r=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i.$$

5 pts. to here. Hence $y_h(t)=c_1e^t\cos(t)+c_2e^t\sin(t)$ is the general solution of the associated homogeneous equation $y''-2y'+2y=0$. To obtain a particular solution of the nonhomogeneous equation, we use the method of undetermined coefficients with

8 pts. to here. a trial form $y_p(t)=A\cos(2t)+B\sin(2t)$. Then $y_p'(t)=-2A\sin(2t)+2B\cos(2t)$

9 pts. to here. and $y_p''(t)=-4A\cos(2t)-4B\sin(2t)$. We want

10 pts. to here. $$y_p''-2y_p'+2y_p=2\cos(2t),$$

so substituting for y_p and its derivatives gives

11 pts. to here. $$-4A\cos(2t)-4B\sin(2t)-2\left[-2A\sin(2t)+2B\cos(2t)\right]+2\left[A\cos(2t)+B\sin(2t)\right]=2\cos(2t).$$

Simplifying yields

12 pts. to here. $$(-2A-4B)\cos(2t)+(-2B+4A)\sin(2t)=2\cos(2t)+0\cdot\sin(2t).$$

13 pts. to here. Comparing like coefficients, we have $-2A-4B=2$ and $-2B+4A=0$.

16 pts. to here. Consequently, $A=-\frac{1}{5}$ and $B=-\frac{2}{5}$, so $y_p(t)=-\frac{1}{5}\cos(2t)-\frac{2}{5}\sin(2t)$.

The general solution of the nonhomogeneous equation is $y=y_h+y_p$ so

17 pts. to here. $$\boxed{y(t)=e^t\left(c_1\cos(t)+c_2\sin(t)\right)-\frac{1}{5}\cos(2t)-\frac{2}{5}\sin(2t)}$$ (c_1, c_2 arbitrary constants).

3.[17] Find the solution of $t^2y''+6ty'+6y=0$ which satisfies the conditions $y(1)=0$ and $y'(1)=10$.

2 pts. to here. $y=t^m$ in $t^2y''+6ty'+6y=0$ leads to $m(m-1)+6m+6=0$ or

5 pts. to here $m^2+5m+6=0$ or $(m+3)(m+2)=0$. Therefore $m=-2$ or $m=-3$ so

11 pts. to here. $y(t)=c_1t^{-2}+c_2t^{-3}$ (c_1, c_2 arbitrary constants) is the general solution of

12 pts. to here. the DE on the interval $t>0$. Note that $y'(t)=-2c_1t^{-3}-3c_2t^{-4}$ so applying the initial conditions gives

14 pts. to here

$$\begin{cases} 0=y(1)=c_1+c_2 \\ 10=y'(1)=-2c_1-3c_2. \end{cases}$$

15 pts. to here. Adding 3 times the first equation to the second yields $10=c_1$ and hence

16 pts. to here. $c_2=-10$. Consequently,

17 pts. to here.

$$\boxed{y(t)=10t^{-2}-10t^{-3}}$$

is the solution of the initial value problem.

4.[17] Recall the definition of the Laplace transform of a function $f = f(t)$ at s:

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st} dt$$

for those values of s for which the improper integral converges.

(a) Use the definition of the Laplace transform to find the Laplace transform of $f(t) = t$.

(b) For which values of s is the Laplace transform of $f(t) = t$ defined?

(a) If $f(t) = t$ then

$$\mathcal{L}\{f\}(s) = \int_0^\infty t e^{-st} dt = \lim_{M \to \infty} \int_0^M t e^{-st} dt.$$

Integrating by parts with $U = t$ and $dV = e^{-st} dt$, we have

$$\mathcal{L}\{f\}(s) = \lim_{M \to \infty} \left(\frac{t e^{-st}}{-s} \Big|_{t=0}^{M} - \int_0^M \frac{e^{-st}}{-s} dt \right)$$

$$= \lim_{M \to \infty} \left(-\frac{M}{s e^{sM}} - \frac{e^{-st}}{s^2} \Big|_{t=0}^{M} \right)$$

$$= \lim_{M \to \infty} \left(\frac{-M}{s e^{sM}} - \frac{1}{s^2 e^{sM}} + \frac{1}{s^2} \right).$$

If $s > 0$, then $\frac{M}{e^{sM}} \to 0$ (by l'Hospital's rule) and $\frac{1}{e^{sM}} \to 0$

as $M \to +\infty$. Therefore

$$\boxed{\mathcal{L}\{f\}(s) = \frac{1}{s^2}}.$$

(b) The Laplace transform of $f(t) = t$ is defined for $\boxed{s > 0}$.

5.[16] Determine the largest interval for which the initial value problem

$$ty''' + \sin(t)y'' + (t^2 - 1)y' + 3y = \tan(t), \quad y(1) = 0,\ y'(1) = 5,\ y''(1) = -1,$$

is sure to have a unique solution. Remember to give reasons for your answer.

We divide through the DE by t to place it in standard form:

1 pt. to here.

$$y''' + \frac{\sin(t)}{t}y'' + \left(\frac{t^2-1}{t}\right)y' + \frac{3}{t}y = \frac{\tan(t)}{t}.$$

The coefficient functions of the DE in standard form are continuous on intervals as follows:

3 pts. to here.

$t \mapsto \frac{\sin(t)}{t}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$;

5 pts. to here.

$t \mapsto \frac{t^2-1}{t}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$;

7 pts. to here.

$t \mapsto \frac{3}{t}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$.

9 pts. to here.

The driver $g(t) = \frac{\tan(t)}{t}$ is continuous on $(-\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, and

10 pts. to here.

$\left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$ for $k = \pm 1, \pm 2, \pm 3, \ldots$ Since the point $t_0 = 1$ in

11 pts. to here.

the initial conditions belongs only to the interval $(0, \frac{\pi}{2})$, the

12 pts. to here

existence-uniqueness theorem for linear initial value problems guarantees

16 pts. to here.

a unique solution on the interval $\boxed{0 < t < \frac{\pi}{2}}$, and this is the largest interval where a unique solution is sure to exist.

6.[16] A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assume that the viscous damping of the fluid and air are so small that they can be safely neglected.

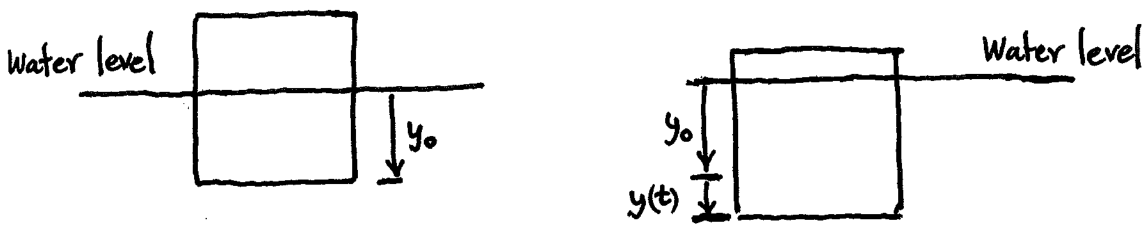
(a) Derive the differential equation of motion of the block.

(b) Determine the period of the block's motion.

You may find Archimedes' principle of use: An object that is completely or partially submerged in a fluid is acted upon by an upward (buoyant) force equal to the weight of the displaced fluid.

(a) Let $y(t)$ denote the vertical displacement of the cube from its static equilibrium position in the fluid at time t.

Side View of Cube



Cube's position at Static Equilibrium

Cube's position at an instant $t > 0$.

Applying Newton's second law of motion and the assumptions on the viscous damping force and the buoyant force on the block, we have

$$m\vec{a} = \vec{F}_{net} = \vec{F}_{gravity} + \vec{F}_{buoyant} + \cancel{\vec{F}_{damping}}^{\;\vec{0}}$$

and thus, with g denoting the (constant) acceleration of gravity,

$$(*) \qquad \rho l^3 y''(t) = \rho l^3 g - \rho_0 l^2 (y(t) + y_0) g.$$

If the block were not displaced initially, then it would remain in its static equilibrium position and $y(t) = y'(t) = 0$ for all t. In this case, $(*)$ would imply $0 = \vec{F}_{net} = \rho l^3 g - \rho_0 l^2 y_0 g$. Substituting this relation into $(*)$, we have

$$\rho l^3 y''(t) = -\rho_0 l^2 g y(t) \quad \text{or equivalently} \quad \boxed{\rho l y'' + \rho_0 g y = 0}.$$

(OVER)

(b) The general solution of $y'' + \frac{\rho_0 g}{\rho \ell} y = 0$ is $y(t) = c_1 \cos\left(\sqrt{\frac{\rho_0 g}{\rho \ell}}\, t\right) + c_2 \sin\left(\sqrt{\frac{\rho_0 g}{\rho \ell}}\, t\right)$, so the period of the cube's motion is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\rho_0 g}{\rho \ell}}} = \boxed{2\pi \sqrt{\frac{\rho \ell}{\rho_0 g}}}\,.$$

2014 Fall Semester, Math 3304 Hour Exam II, Master List

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Number taking exam: 443
Median: 76
Mean: 72.3
Standard Deviation: 15.7

Number receiving A's: 33 7.4 %
Number receiving B's: 145 32.7
Number receiving C's: 107 24.2
Number receiving D's: 78 17.6
Number receiving F's: 80 18.1