Math 3304 Fall 2015 Exam 3

Your printed name: Solutions

Your instructor's name: ______

Your section (or Class Meeting Days and Time): ______

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noise making devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 3 consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] Solve the initial value problem

$$y'' + 3y' + 2y = u_{10}(t)$$

subject to $y(0) = 0,\ y'(0) = 0.$

#10 $s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 3sY(s) - 3\cancel{y(0)}^{0} + 2Y(s) = \frac{e^{-10s}}{s}$ #11

$$(s^2 + 3s + 2)Y(s) = \frac{e^{-10s}}{s}$$

$$Y(s) = \frac{e^{-10s}}{s(s+1)(s+2)}$$

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + C(s+1)s$$

$s = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$s = -1 \Rightarrow 1 = -B \Rightarrow B = -1$

$s = -2 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

$$\therefore\ Y(s) = e^{-10s}\left[\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right]$$

$$y(t) = u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]$$ #12

#2 #2

2. [20] Find the solution of the integral equation

$$y(t) - \underbrace{\int_0^t y(\tau)\sin(t-\tau)d\tau}_{y(t) * \sin t} = \delta(t-\pi)\cos t.$$

$$\overset{\#14}{Y(s) - \frac{1}{s^2+1} Y(s)} = \mathcal{L}\{\delta(t-\pi)\cos t\} \overset{\text{defn.}}{=} \cos\pi \cdot e^{-\pi s}$$

#4

$$\frac{s^2}{s^2+1} Y(s) = -e^{-\pi s} \Rightarrow Y(s) = -e^{-\pi s} \cdot \frac{s^2+1}{s^2}$$

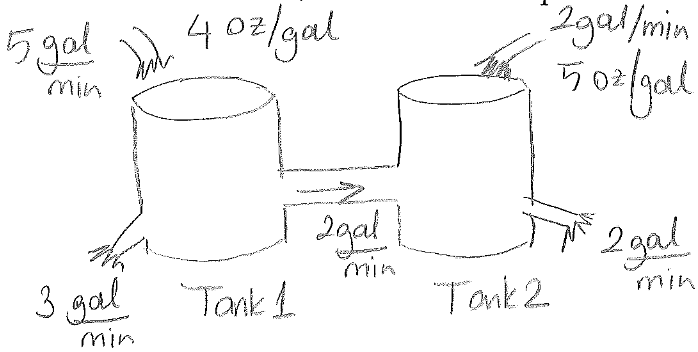
$$Y(s) = -e^{-\pi s}\left[1 + \frac{1}{s^2}\right]$$

#3

$$y(t) = -\delta(t-\pi) - u_\pi(t)\cdot(t-\pi)$$

#15 #12

3. [20] Consider two tanks holding initially 100 gallons of pure water each. A mixture of salt and water at a concentration of 4 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture in Tank 1 drains into the environment at a rate of 3 gallons per minute, and from Tank 1 into Tank 2 at a rate 2 gallons per minute. Another mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 2 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, the differential equations and initial conditions that model the flow process.



$$V_1(t) = V_1(0) = 100 \text{ gal}$$

$$\frac{dV_2}{dt} = 2+2-2 = 2 \text{ gal/min}$$

$$V_2 = 2t + C, \quad V_2(0) = 100$$

$$\therefore V_2(t) = 100 + 2t$$

Net Rate = Rate In − Rate Out

$$\frac{dQ_1}{dt} = \left(4\frac{\text{oz}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) - \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right)$$

$$\frac{dQ_2}{dt} = \left(5\frac{\text{oz}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) + \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right) - \left(\frac{Q_2(t)\text{ oz}}{V_2(t)\text{ gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right)$$

$$\therefore Q_1' = 20 - \frac{Q_1}{20}, \quad Q_1(0) = 0.$$

$$Q_2' = 10 + \frac{Q_1}{50} - \frac{Q_2}{50+t}, \quad Q_2(0) = 0.$$

4. [20] Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

$$\begin{vmatrix} -\lambda & -4 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \mp 2i$$

$$\lambda = 2i \Rightarrow \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \xi_1 - 2i\,\xi_2 = 0$$

$$\Rightarrow \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$x^{(1)} = \xi e^{\lambda t} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} \left(\cos 2t + i \sin 2t\right)$$

$$= \underbrace{\begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}}_{u} + i \underbrace{\begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}}_{v}$$

$\therefore$ The general soln. is

$$x = c_1 \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$$

5. [20] Given that $\Phi(t) = \begin{pmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{pmatrix}$ is a fundamental matrix for

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$X = \Phi(t) \cdot c$ is the general soln. of $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x$.

$$x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$2 = c_1 + \cancel{c_2}$$
$$+1 = -3c_1 - \cancel{c_2}$$

$$c_1 = -\frac{3}{2} \Rightarrow c_2 = 2 - c_1 = 2 + \frac{3}{2} = \frac{7}{2}.$$

$$\therefore \quad X = \begin{pmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{pmatrix} \begin{pmatrix} -3/2 \\ 7/2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} e^{-t} + \frac{7}{2} e^{t} \\ -\frac{9}{2} e^{-t} + \frac{7}{2} e^{t} \end{pmatrix}$$

OR

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
10.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
11.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
12.	$u_c(t)\, f(t-c)$	$e^{-cs} F(s)$
13.	$e^{ct} f(t)$	$F(s-c)$
14.	$(f * g)(t)$	$F(s)G(s)$
15.	$\delta(t-c)$	e^{-cs}