Mathematics 3304

Fall 2014

Final Exam

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 9 pages of problems containing 9 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown for all integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [22] at the beginning of a problem indicates the point value of that problem is 22. The maximum possible score on this exam is 220.

problem	1	2	3	4	5	6	7	8	9	**Sum**
points earned										
maximum points	22	22	22	22	22	33	33	22	22	220

1.[22] Find the explicit solution the differential equation $\frac{1}{\sqrt{t}}+\sqrt{y}y'=t$.

The DE is first order and separable.

$$y^{1/2}\frac{dy}{dt} = t - t^{-1/2}$$

$$\int y^{\frac{1}{2}}dy = \int(t-t^{-\frac{1}{2}})\,dt$$

$$\frac{2}{3}y^{\frac{3}{2}} = \frac{1}{2}t^{2} - 2t^{\frac{1}{2}} + c_1$$

$$y^{\frac{3}{2}} = \frac{3}{4}t^{2} - 3t^{\frac{1}{2}} + c \qquad \left(\frac{3}{2}c_1 = c\right)$$

$$\boxed{y(t) = \left(\frac{3}{4}t^{2} - 3t^{1/2} + c\right)^{2/3}}$$

2.[22] Solve the initial value problem $\frac{1}{t}y' - \frac{2}{t^2}y = t\cos(t)$, $y(\pi) = \frac{\pi^2}{2}$.

The DE is linear and of first order. Normalizing we have

(*) $y' - \frac{2}{t}y = t^2\cos(t)$.

An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{2}{t}dt} = e^{-2\ln(t) + \cancel{c}^{0}} = t^{-2}$.

Multiplying both sides of (*) by the integrating factor yields

$$t^{-2}y' - 2t^{-3}y = \cos(t).$$

But the LHS of this last DE is exact: $(t^{-2}y)' = t^{-2}y' - 2t^{-3}y$. Therefore

$$(t^{-2}y)' = \cos(t).$$

Integrating both sides, we obtain

$$t^{-2}y = \int \cos(t)dt = \sin(t) + c.$$

Multiplying both sides by t^2, we isolate y:

$$y(t) = t^2\sin(t) + ct^2.$$

We need to choose the arbitrary constant c so the initial condition is satisfied:

$$\frac{\pi^2}{2} = y(\pi) = \pi^2\underbrace{\sin(\pi)}_{0} + c\pi^2 = c\pi^2.$$

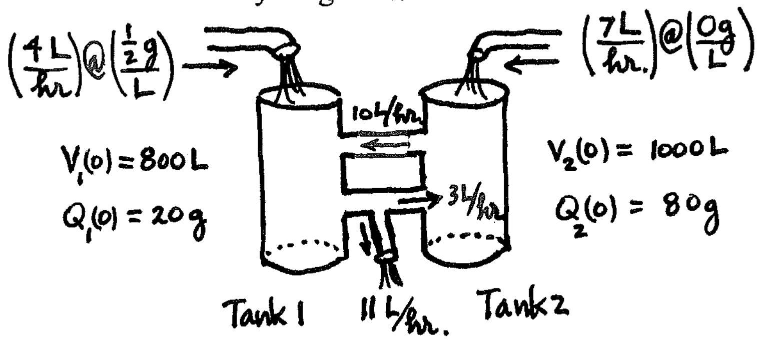
Therefore $c = \frac{1}{2}$. The solution is

$$\boxed{y(t) = t^2\sin(t) + \frac{1}{2}t^2}.$$

3. Two 1000 liter tanks contain well-stirred salt water solutions. Tank 1 initially contains 800 liters of water and 20 grams of dissolved salt and Tank 2 contains 1000 liters of water and 80 grams of dissolved salt. Salt water with a concentration of ½ grams per liter of salt enters Tank 1 at a rate of 4 liters per hour. Fresh water enters Tank 2 at a rate of 7 liters per hour. Solution flows from Tank 2 into Tank 1 through a connecting pipe at a rate of 10 liters per hour. Through a different connecting pipe, 14 liters of solution per hour flows out of Tank 1, of which 3 liters per hour flows into Tank 2 and the remaining 11 liters per hour flows completely out of the system.

(a) [18] Set up, **BUT DO NOT SOLVE**, an initial value problem which models the amount of salt in each tank at all future times.

(b) [4] Determine the amount of salt in each tank at equilibrium, that is, the amount of salt in each tank after a very long time.



Let $Q_1(t)$ denote the number of grams of salt in Tank 1 at time t hours.
Let $Q_2(t)$ denote the number of grams of salt in Tank 2 at time t hours.
We apply the principle

Net Rate = Rate In − Rate Out

to each tank.

Tank 1: $$\frac{dQ_1}{dt} = \left(\frac{4\cancel{L}}{hr}\right)\left(\frac{\frac{1}{2}g}{\cancel{L}}\right) + \left(\frac{10\cancel{L}}{hr}\right)\left(\frac{Q_2(t)\,g}{V_2(t)\cancel{L}}\right) - \left(\frac{14\cancel{L}}{hr}\right)\left(\frac{Q_1(t)\,g}{V_1(t)\cancel{L}}\right), \quad Q_1(0) = 20g$$

Tank 2: $$\frac{dQ_2}{dt} = \left(\frac{7\cancel{L}}{hr}\right)\left(\frac{0g}{\cancel{L}}\right) + \left(\frac{3\cancel{L}}{hr}\right)\left(\frac{Q_1(t)\,g}{V_1(t)\cancel{L}}\right) - \left(\frac{10\cancel{L}}{hr}\right)\left(\frac{Q_2(t)\,g}{V_2(t)\cancel{L}}\right), \quad Q_2(0) = 80g$$

Note that 14 L/hr of solution is flowing into Tank 1 and 14 L/hr of solution is flowing out of Tank 1 so the volume of solution in Tank 1 is constant: $V_1(t) = V_1(0) = 800L$. A similar analysis shows $V_2(t) = V_2(0) = 1000L$. Therefore, the IVP modeling the system is

(a) $$\boxed{\begin{aligned} Q_1' &= -\frac{7}{400}Q_1 + \frac{1}{100}Q_2 + 2, & Q_1(0) &= 20, \\ Q_2' &= \frac{3}{800}Q_1 - \frac{1}{100}Q_2, & Q_2(0) &= 80. \end{aligned}}$$

(b) The system in (a) can be expressed as $\vec{Q}' = A\vec{Q} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ where $\vec{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ and $A = \begin{bmatrix} -\frac{7}{400} & \frac{1}{100} \\ \frac{3}{800} & -\frac{1}{100} \end{bmatrix}$. At equilibrium, the amounts of salt no longer change so $\vec{Q}' = \vec{0}$.

Thus $\vec{0} = A\vec{Q}_e + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ so $$\vec{Q}_e = -A^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = -\frac{1}{11/80{,}000}\begin{bmatrix} -\frac{1}{100} & -\frac{1}{100} \\ -\frac{3}{800} & -\frac{7}{400} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1600}{11} \\ \frac{600}{11} \end{bmatrix}.$$

Therefore Tank 1 contains $\frac{1600}{11} = 145.\overline{45}$ g and Tank 2 contains $\frac{600}{11} = 54.\overline{54}$ g of salt at equilibrium.

$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \ln(t)$ so $g(t) = \ln(t)$.

4. [22] Solve the differential equation $t^2y'' - 3ty' + 4y = t^2\ln(t)$ on the interval $t > 0$.

The DE is a second order, linear, nonhomogeneous Euler equation. We let $y = t^m$ in $t^2y'' - 3ty' + 4y = 0$ to obtain $m(m-1) - 3m + 4 = 0$. Simplifying,

$$0 = m^2 - 4m + 4 = (m-2)^2 \quad \text{so} \quad m = 2 \text{ with multiplicity two.}$$

Therefore $y_h(t) = c_1t^2 + c_2t^2\ln(t)$.

We use variation of parameters to find a particular solution to the nonhomogeneous equation.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^2\ln(t) \\ 2t & 2t\ln(t) + t^2\cdot\frac{1}{t} \end{vmatrix} = t^3 \neq 0 \quad \text{on } t > 0.$$

Then

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Let $v = \ln t$

where $u_1(t) = \int \frac{-g(t)y_2(t)}{W(y_1,y_2)(t)}\,dt = \int -\frac{\ln(t)\cdot t^2\ln(t)}{t^3}\,dt = -\int (\ln t)^2\cdot\frac{1}{t}\,dt = -\int v^2\,dv$

$$= -\frac{1}{3}v^3 + \cancel{c}^{\,0} = -\frac{1}{3}\ln^3(t),$$

Let $v = \ln t$

and $u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)(t)}\,dt = \int \frac{\ln(t)\cdot t^2}{t^3}\,dt = \int \ln(t)\cdot\frac{1}{t}\,dt = \int v\,dv = \frac{1}{2}v^2 + \cancel{c}^{\,0}$

$$= \frac{1}{2}\ln^2(t).$$

Therefore $y_p(t) = \left(-\frac{1}{3}\ln^3(t)\right)t^2 + \left(\frac{1}{2}\ln^2(t)\right)t^2\ln(t) = \frac{1}{6}t^2\ln^3(t)$.

Consequently, the general solution on the interval $t > 0$ is

$$y = y_h + y_p$$

or

$$\boxed{y(t) = c_1t^2 + c_2t^2\ln(t) + \frac{1}{6}t^2\ln^3(t)}$$

where c_1 and c_2 are arbitrary constants.

5. [22] Find the general solution of the differential equation $y^{(4)} - 16y = \cos(t)$.

The DE is fourth order, linear, and nonhomogeneous with constant coefficients. We let $y = e^{rt}$ in $y^{(4)} - 16y = 0$ to obtain $r^4 - 16 = 0$. Simplifying,

$$0 = r^4 - 16 = (r^2-4)(r^2+4) = (r-2)(r+2)(r^2+4),$$

so the roots are $2, -2, 2i$, and $-2i$. Consequently,

$$y_h(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(2t) + c_4\sin(2t).$$

We use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. Since $g(t) = \cos(t)$ and this is not a solution to the homogeneous equation, a trial form is $y_p(t) = A\cos(t) + B\sin(t)$ where A and B are constants to be determined. Then

$$y_p'(t) = -A\sin(t) + B\cos(t), \quad y_p''(t) = -A\cos(t) - B\sin(t), \quad y_p'''(t) = A\sin(t) - B\cos(t)$$

and $y_p^{(4)}(t) = A\cos(t) + B\sin(t)$. We want

$$y_p^{(4)} - 16y_p = \cos(t)$$

so substituting yields

$$A\cos(t) + B\sin(t) - 16\big(A\cos(t) + B\sin(t)\big) = \cos(t)$$

or

$$-15A\cos(t) - 15B\sin(t) = 1\cdot\cos(t) + 0\cdot\sin(t)$$

Therefore $-15A = 1$ and $-15B = 0$ so $y_p(t) = -\frac{1}{15}\cos(t)$.

The general solution is

$$y = y_h + y_p$$

or

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(2t) + c_4\sin(2t) - \frac{1}{15}\cos(t)}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

6. (a) [4] Express the function

$$g(t)=\begin{cases} t & \text{if } 0\le t<1, \\ 1 & \text{if } 1\le t<\infty, \end{cases}$$

in terms of the unit step function.

(b) [25] Solve the initial value problem $y''+y=t-u_1(t)(t-1)$, $y(0)=0=y'(0)$.

(c) [4] Evaluate the solution to part (b) at the two points $t=\pi/4$ and $t=\pi$.

(a) $$g(t) = t\cdot\underbrace{(1-u_1(t))}_{\text{off switch at 1}} + 1\cdot\underbrace{u_1(t)}_{\text{on switch at 1}} = \boxed{t-(t-1)u_1(t)}.$$

(b) We use the Laplace transform method to solve the IVP. Taking the Laplace transform of both sides of the DE and using linearity plus formulas 6, 2, and 8 in the Laplace transform table gives

$$\mathcal{L}\{y''\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{t\}(s) - \mathcal{L}\{(t-1)u_1(t)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + \mathcal{L}\{y\}(s) = \frac{1}{s^2} - e^{-s}\cdot\frac{1}{s^2}.$$

Applying the initial conditions and solving for $\mathcal{L}\{y\}(s)$ yields

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s^2(s^2+1)}.$$

A partial fraction decomposition calculation leads to

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+1} \quad \text{so} \quad 1 = As(s^2+1)+B(s^2+1)+(Cs+D)s^2.$$

Setting $s=0$ gives $1=B$. Setting $s=i$ gives $1=-(Ci+D)$ or $1+0i=-D-Ci$. Therefore $D=-1$ and $C=0$. Setting $s=1$ gives $1=2A+2B+C+D$. Substituting the known values of $B=1, C=0,$ and $D=-1$ gives $1=2A+1$ so $A=0$. Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-s}\cdot\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)\right\}$$

or $\boxed{y(t) = t-\sin(t) - u_1(t)(t-1-\sin(t-1))}$ by formulas 2, 3, and 8 in the Laplace transform table.

(c) $$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \sin\left(\frac{\pi}{4}\right) - \cancel{\overset{0}{u_1\left(\frac{\pi}{4}\right)}}\left(\frac{\pi}{4}-1-\sin\left(\frac{\pi}{4}-1\right)\right) = \boxed{\frac{\pi}{4}-\frac{\sqrt{2}}{2}}$$

$$y(\pi) = \pi - \underbrace{\sin(\pi)}_{0} - \cancel{\overset{1}{u_1(\pi)}}(\pi-1-\sin(\pi-1)) = \cancel{\pi} - (\cancel{\pi}-1-\sin(\pi-1)) = \boxed{1+\sin(\pi-1)}.$$

7.[33] Solve the initial value problem $y'(t)+2y(t)=\int_0^t 2\sin(t-\xi)\,y(\xi)\,d\xi, \quad y(0)=1.$

Since $(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau$, we recognize $\int_0^t 2\sin(t-\xi)y(\xi)\,d\xi$ as the convolution product of $f(t)=2\sin(t)$ and $g(t)=y(t)$. Rewriting the integro-differential equation, we have

$$y'(t)+2y(t)=(2\sin * y)(t).$$

Taking the Laplace transform of both sides of this equation and using linearity and formulas 6 and 5 in the Laplace transform table yields

$$\mathcal{L}\{y'\}(s)+2\mathcal{L}\{y\}(s)=2\mathcal{L}\{\sin * y\}(s)$$

$$s\mathcal{L}\{y\}(s)-y(0)+2\mathcal{L}\{y\}(s)=2\mathcal{L}\{\sin(t)\}(s)\cdot\mathcal{L}\{y\}(s)$$

$$(s+2)\mathcal{L}\{y\}(s)-1=2\left(\frac{1}{s^2+1}\right)\mathcal{L}\{y\}(s).$$

Rearranging gives

$$\left(s+2-\frac{2}{s^2+1}\right)\mathcal{L}\{y\}(s)=1.$$

Multiplying through by s^2+1, we have

$$\left((s+2)(s^2+1)-2\right)\mathcal{L}\{y\}(s)=s^2+1$$

$$(s^3+2s^2+s+\cancel{2}-\cancel{2})\mathcal{L}\{y\}(s)=s^2+1.$$

Solving for $\mathcal{L}\{y\}(s)$ yields

$$\mathcal{L}\{y\}(s)=\frac{s^2+1}{s^3+2s^2+s}=\frac{s^2+1}{s(s^2+2s+1)}=\frac{s^2+1}{s(s+1)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}.$$

But then $s^2+1=A(s+1)^2+Bs(s+1)+Cs$. Setting $s=0$ gives $1=A$. Setting $s=-1$ gives $2=-C$. Setting $s=1$ gives $2=4A+2B+C$ and substituting the known values $A=1$ and $C=-2$, we have $2=2+2B$ so $B=0$. Therefore

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{2}{(s+1)^2}\right\}=1-2e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=\boxed{1-2te^{-t}}$$

by formulas 2 and 7 in the Laplace transform table.

8.[22] Solve the system $\mathbf{x}' = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix} \mathbf{x}$, subject to the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $A = \begin{bmatrix} 6 & -4 \\ 1 & 2 \end{bmatrix}$. Then $\vec{x}(t) = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$, the eigenvalue equation for the matrix A. The eigenvalues r of A satisfy

$$0 = \det(A - rI) = \begin{vmatrix} 6-r & -4 \\ 1 & 2-r \end{vmatrix} = (6-r)(2-r) + 4 = r^2 - 8r + 16 = (r-4)^2.$$

Therefore $r = 4$ with multiplicity two. An eigenvector $\vec{k}$ of A corresponding to $r = 4$ satisfies $(A - 4I)\vec{k} = \vec{0}$ or $\begin{bmatrix} 6-4 & -4 \\ 1 & 2-4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{cases} 2k_1 - 4k_2 = 0 \\ k_1 - 2k_2 = 0 \end{cases}$

Notice that the first equation of the last system is 2 times the second equation, so the first equation is redundant. Consequently, the solution of the system is $k_1 = 2k_2$ where k_2 is arbitrary: $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Take $k_2 = 1$ for convenience.

Thus $\vec{x}^{(1)}(t) = \vec{k}e^{4t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$ solves $\vec{x}' = A\vec{x}$.

A second solution has the form $\vec{x}^{(2)}(t) = \vec{k}te^{rt} + \vec{l}e^{rt}$ where $\vec{k}, \vec{l}$, and r are constants. Substituting $\vec{x} = \vec{x}^{(2)}(t)$ in $\vec{x}' = A\vec{x}$ leads to

$$\begin{cases} (A - rI)\vec{k} = 0, \\ (A - rI)\vec{l} = \vec{k}. \end{cases}$$

We know that $r = 4$ and $\vec{k} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ solve the first equation from previous work. Therefore the second equation becomes $(A - 4I)\vec{l} = \vec{k}$ or $\begin{bmatrix} 6-4 & -4 \\ 1 & 2-4 \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

This is equivalent to $\begin{cases} 2l_1 - 4l_2 = 2, \\ l_1 - 2l_2 = 1. \end{cases}$ As before, the first equation is twice the second,

so $l_1 = 1 + 2l_2$ (where l_2 is arbitrary) solves $(A - 4I)\vec{l} = \vec{k}$. Taking $l_2 = 0$ for convenience, we find $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consequently, $\vec{x}^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t}$ is a second solution of $\vec{x}' = A\vec{x}$. The general solution is $\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$. To satisfy the initial condition: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}(0) = c_1\vec{x}^{(1)}(0) + c_2\vec{x}^{(2)}(0) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $c_1 = 1$ and $c_2 = -1$.

Thus, the solution to the IVP is

$$\vec{x}(t) = 1 \cdot \vec{x}^{(1)}(t) - 1 \cdot \vec{x}^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} - \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t} \right) = \boxed{\begin{bmatrix} 1-2t \\ 1-t \end{bmatrix} e^{4t}}.$$

9. [22] Let $A=\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$ and $\mathbf{g}(t)=\begin{pmatrix}e^t\\ -e^t\end{pmatrix}$. Given that $\mathbf{\Psi}(t)=\begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}$ is a fundamental matrix for the system $\mathbf{x}'=A\mathbf{x}$, find the general solution of the nonhomogeneous system $\mathbf{x}'=A\mathbf{x}+\mathbf{g}(t)$.

The general solution to $\vec{x}'=A\vec{x}$ is $\vec{x}_h(t)=\Psi(t)\vec{c} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t}$. To find a particular solution of $\vec{x}'=A\vec{x}+\vec{g}(t)$, we use the variation of parameters formula:

$$\vec{x}_p(t) = \Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds.$$

$$\Psi^{-1}(s) = \begin{bmatrix}e^s & e^{-s}\\ e^s & 3e^{-s}\end{bmatrix}^{-1} = \frac{1}{\det\Psi(s)}\begin{bmatrix}\psi_{22}(s) & -\psi_{12}(s)\\ -\psi_{21}(s) & \psi_{11}(s)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3e^{-s} & -e^{-s}\\ -e^{s} & e^{s}\end{bmatrix}.$$

$$\int^t \Psi^{-1}(s)\vec{g}(s)\,ds = \int^t \frac{1}{2}\begin{bmatrix}3e^{-s} & -e^{-s}\\ -e^{s} & e^{s}\end{bmatrix}\begin{bmatrix}e^s\\ -e^s\end{bmatrix}ds = \int^t \frac{1}{2}\begin{bmatrix}3+1\\ -1-e^{2s}\end{bmatrix}ds = \begin{bmatrix}2t\\ -\frac{t}{2}-\frac{e^{2t}}{4}\end{bmatrix}$$

$$\therefore\quad \vec{x}_p(t) = \begin{bmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{bmatrix}\begin{bmatrix}2t\\ -\frac{t}{2}-\frac{e^{2t}}{4}\end{bmatrix} = 2te^t\begin{bmatrix}1\\1\end{bmatrix} + \left(-\frac{t}{2}-\frac{e^{2t}}{4}\right)e^{-t}\begin{bmatrix}1\\3\end{bmatrix}.$$

The general solution of $\vec{x}'=A\vec{x}+\vec{g}(t)$ is

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

or

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} + 2te^t\begin{bmatrix}1\\1\end{bmatrix} - \left(\frac{te^{-t}}{2}+\frac{e^t}{4}\right)\begin{bmatrix}1\\3\end{bmatrix}}$$

where c_1 and c_2 are arbitrary constants.

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f * g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}