Math 204 Spring 2011
Exam I

Your printed name: Solutions

Your instructor's name: ______

Your section (or Class Meeting Days and Time): ______

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) for the duration of the exam.
3. Exam I consists of this cover page, and 6 pages of problems containing 6 numbered problems.
4. Once the exam begins, you will have 60 minutes to complete your answers.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	6	Sum
Points Earned							
Max. Points	16	16	16	18	16	18	100

1. [16] Consider, **BUT DO NOT SOLVE**, the differential equation $y' = 2y\,(9 - y^2)$.

   a) Determine the equilibrium solutions (critical points) of the differential equation.

   b) Sketch a phase line and/or typical solution curves in the $t - y$ plane. **Justify your answer. "My calculator says so" is NOT a valid justification.**

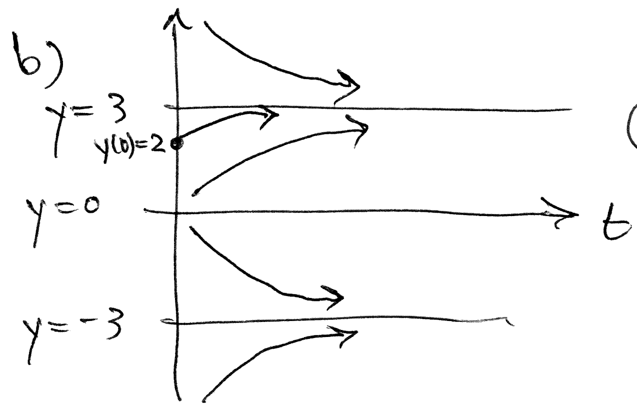
   c) Use your answer from part (b) to classify each critical point as either asymptotically stable, unstable, or semi-stable.

   d) If $y(t)$ denotes the solution of the differential equation satisfying the initial condition $y(0) = 2$, determine $\lim_{t\to\infty} y(t)$.

a) $y' = 2y(9-y^2) = 2y(3-y)(3+y) = 0$

3 $\Rightarrow \underline{y = 0,\ 3,\ -3 \text{ eq solns}}$

b)



5

	$2y$	$(3-y)$	$(3+y)$	$= y'$
$y=4$	+	−	+	−
$y=1$	+	+	+	+
$y=-1$	−	+	+	−
$y=-4$	−	+	−	+

c) Plots of soln curves in (b) show

2 $y = 3$ is asy. stable

2 $y = 0$ is unstable

2 $y = -3$ is asy. stable

2 d) $\lim_{t\to\infty} y(t) = \underline{3}$ when $y(0) = 2$ by parts b) & c)

2. [16] Find the general solution of $y' + \frac{y}{t} = \sin(3t)$. (for $t > 0$)

$$IF = e^{\int \frac{1}{t}\,dt} = e^{\ln|t|} = |t| = t$$

Multiply by IF: $t y' + y = t \sin(3t)$

product rule $\Rightarrow$ $(t y)' = t \sin(3t)$

check: $(ty)' = ty' + y$ ✓

integrate: $\int (ty)'\,dt = \int \underbrace{t}_{u} \underbrace{\sin(3t)\,dt}_{dv}$

$du = dt \qquad v = \frac{-1}{3}\cos(3t)$

$$\Rightarrow ty = uv - \int v\,du$$

$$\Rightarrow ty = \frac{-1}{3} t \cos(3t) - \int \frac{-1}{3}\cos(3t)\,dt$$

$$\Rightarrow ty = \frac{-1}{3} t \cos(3t) + \frac{1}{9}\sin(3t) + C$$

$$\Rightarrow y = \frac{C}{t} - \frac{1}{3}\cos(3t) + \frac{1}{9t}\sin(3t)$$

3. [16] Find the explicit solution of the initial value problem

$$y' = \frac{x^2+5x+3}{2(y+1)}, \quad y(0) = -2.$$

$$2(y+1)\,dy = (x^2+5x+3)\,dx \qquad \left(\text{for } y' = \frac{dy}{dx}\right)$$

$$\Rightarrow \int 2y+2\,dy = \int x^2+5x+3\,dx$$

$$\Rightarrow y^2+2y = \frac{1}{3}x^3+\frac{5}{2}x^2+3x+C$$

$y(0)=-2$ $(x=0,\ y=-2)$ $\Rightarrow (-2)^2+2(-2) = \frac{1}{3}(0)^3+\frac{5}{2}(0)^2+3(0)+C$

$$\Rightarrow 4-4=0=C$$

$$\Rightarrow y^2+2y = \frac{1}{3}x^3+\frac{5}{2}x^2+3x$$

Approach 1: Complete the square $y^2+2y\underbrace{+1-1}_{=0} = (y+1)^2-1$

$$\Rightarrow (y+1)^2-1 = \frac{1}{3}x^3+\frac{5}{2}x^2+3x$$

$$\Rightarrow (y+1)^2 = \frac{1}{3}x^3+\frac{5}{2}x^2+3x+1$$

$$\Rightarrow y+1 = \pm\sqrt{\frac{1}{3}x^3+\frac{5}{2}x^2+3x+1}$$

must take $-$ since $y(0)=-2$

$$\Rightarrow \underline{y = -1-\sqrt{\frac{1}{3}x^3+\frac{5}{2}x^2+3x+1}}$$

Approach 2: Quadratic formula $y^2+2y-\left(\frac{1}{3}x^3+\frac{5}{2}x^2+3x\right)=0$

$$\Rightarrow y = \frac{-2\pm\sqrt{4-(4)(1)\left[-\left(\frac{1}{3}x^3+\frac{5}{2}x^2+3x\right)\right]}}{2}$$

factor 4 out of square root $\Rightarrow \underline{y = -1-\sqrt{\frac{1}{3}x^3+\frac{5}{2}x^2+3x+1}}$

for initial condition as above

4. a) [8] Let $P(t)$ denote the population of fish in a certain lake at time t. Suppose the birth rate of the fish is twice the current fish population and the death rate of the fish equals the square of the current fish population. Also suppose that the fish are harvested at a constant rate h. Write down, but **DO NOT SOLVE**, a differential equation that models the fish population $P(t)$.

change in pop. = rate pop. added (birth) − rate pop. subtracted (death & harvesting)

$$\Rightarrow \quad P'(t) = 2P(t) - \left([P(t)]^2 + h \right)$$

$$\Rightarrow \quad \underline{P'(t) = 2P(t) - [P(t)]^2 - h}$$

b) [10] A tank initially contains 200 liters of pure water. A mixture of salt and water containing 10 grams per liter of salt enters the tank at a rate of 3 liters per minute, and the well-stirred mixture leaves the tank at a rate of 4 liters per minute. Set up, but **DO NOT SOLVE**, an initial value problem that models the amount $A(t)$ of salt in the tank at any time t.

change in amount of salt = rate salt in − rate salt out

$$\Rightarrow \quad A'(t) = \left(\frac{10\,g}{1\,L} \right) \left(\frac{3\,L}{1\,min} \right) - \left(\frac{A(t)\,g}{V(t)\,L} \right) \left(\frac{4\,L}{1\,min} \right)$$

$$V(t) = \text{volume at time } t = 200 + \underbrace{3t}_{\text{amount added/min}} - \underbrace{4t}_{\text{amount leaving per minute}} = 200 - t$$

$$\Rightarrow \quad \underline{A' = 30 - \frac{4A}{200 - t}} \quad \text{and} \quad \underline{A(0) = 0}$$

pure water = no salt

5. a) [8] Find the general solution of $y'' + 4y' + y = 0$.

$$y = e^{rt} \Rightarrow r^2 + 4r + 1 = 0$$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

$$\Rightarrow \underline{y = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}}$$

b) [8] Find the general solution of $4y'' + 12y' + 9y = 0$.

$$y = e^{rt} \Rightarrow 4r^2 + 12r + 9 = 0$$

$$\Rightarrow r^2 + 3r + \frac{9}{4} = 0$$

$$\Rightarrow (r + 3/2)^2 = 0 \quad \left(\text{or use quadratic formula}\right)$$

$$\Rightarrow r = -3/2 \text{ repeated}$$

$$\Rightarrow \underline{y = c_1 e^{-3/2 t} + c_2 t e^{-3/2 t}}$$

6. [18] Given that $y_1(t) = t^{-1}$ is a solution of the differential equation $t^2 y'' + 3ty' + y = 0$ for $t > 0$, use reduction of order to find a second solution that is not a constant multiple of y_1.

$$y_2 = u y_1 = t^{-1} u$$

$$y_2' = -t^{-2} u + t^{-1} u' \quad \leftarrow \text{prod. rule}$$

$$y_2'' = (2t^{-3} u - t^{-2} u') + (-t^{-2} u' + t^{-1} u'')$$

Sub into $t^2 y'' + 3ty' + y = 0$

$$\Rightarrow t^2 [2t^{-3} u - 2t^{-2} u' + t^{-1} u'']$$

$$+ 3t[-t^{-2} u + t^{-1} u'] + [t^{-1} u] = 0$$

$$\Rightarrow u''(t) + u'(-2+3) + u\underbrace{(2t^{-1} - 3t^{-1} + t^{-1})}_{=0} = 0$$

$$\Rightarrow t u'' + u' = 0$$

Let $v = u'$ $\Rightarrow$ $tv' + v = 0$

$$\Rightarrow (tv)' = 0$$

can use IF to show this or solve by sep. of vars.

$$\Rightarrow tv = C$$

$$\Rightarrow tu' = C \Rightarrow u' = \frac{C}{t}$$

$$\Rightarrow \int u' \, dt = \int \frac{C}{t} \, dt = C \ln t + D$$

$$\Rightarrow u = C \ln t + D \qquad \text{take } C = 1, D = 0$$

$$\Rightarrow u = \ln t$$

$$\Rightarrow y_2 = t^{-1} u = t^{-1} \ln t$$