Math 204 Spring 2011
Exam II

Your printed name: Solutions

Your instructor's name: ____________________

Your section (or Class Meeting Days and Time): ____________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) for the duration of the exam.
3. Exam II consists of this cover page, 5 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.
4. Once the exam begins, you will have 60 minutes to complete your answers.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	6	Sum
Points Earned							
Max. Points	13	13	14	20	20	20	100

1. [13] A body weighing 64 pounds stretches a spring 2 feet beyond its natural length. The body moves in a medium that impacts a viscous damping force of 8 pounds when the speed of the body is 3 feet per second. The body is pulled down an additional 1 foot and is set in motion with an initial upward velocity of 5 feet per second. Set up, but **DO NOT SOLVE**, an initial value problem that models the motion of the body.

Note: Use 32 ft/sec^2 as the acceleration of gravity in this problem.

$$m x'' + \gamma x' + k x = f(t)$$

$$W = mg \Rightarrow 64 = m(32) \Rightarrow \underline{m = 2}$$

$$\underline{\underline{\text{IVP}}} \quad \Rightarrow \left[\begin{array}{l} 2x'' + \frac{8}{3}x' + 32x = 0 \\ x(0) = 1 \\ x'(0) = -5 \end{array}\right]$$

minus sign since down is +

no external force

$$W = k(\text{stretch}) \Rightarrow 64 = k(2) \Rightarrow \underline{k = 32}$$

$$\text{damping force} = \gamma x' \Rightarrow 8 = \gamma(3) \Rightarrow \underline{\gamma = 8/3}$$

2. [13] Find the general solution of the differential equation $x^2 y'' + 7xy' + 13y = 0$, for $x > 0$.

$$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$$

$$\Rightarrow x^2\left[m(m-1)x^{m-2}\right] + 7x\left[m x^{m-1}\right] + 13\left[x^m\right] = 0$$

$$\Rightarrow m(m-1) + 7m + 13 = 0 \qquad (\text{divided out } x^m)$$

$$\Rightarrow m^2 + 6m + 13 = 0$$

$$\Rightarrow m = \frac{-6 \pm \sqrt{36 - 4(13)}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$$

$$\Rightarrow \underline{y = C_1 x^{-3}\cos(2\ln x) + C_2 x^{-3}\sin(2\ln x)}$$

3. [14] Find the inverse Laplace transform of $\dfrac{3s+5}{s^2-4s+5}$.

Complete the square

$$s^2-4s+5 = s^2-4s+\left(\frac{-4}{2}\right)^2-\left(\frac{-4}{2}\right)^2+5$$

$$= (s-2)^2+1$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3s+5}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+5}{(s-2)^2+1}\right\}$$

Next: $3s+5 = (3s-6)+(6+5) = 3(s-2)+11$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-2)+11}{(s-2)^2+1}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1^2}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1^2}\right\}$$

$$= \underline{3e^{2t}\cos t + 11e^{2t}\sin t}$$

(item ~~7~~ 6 in table with $a=2$, $b=1$) (item 6 in table with $a=2$, $b=1$)

4. [20] Find the general solution of $y'' + 4y' + 4y = t^{-2}e^{-2t}$, for $t > 0$.

First, solve $y_h'' + 4y_h' + 4y_h = 0$.

$y_h = e^{rt} \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$

$\Rightarrow r = -2$ repeated $\Rightarrow \underline{y_h = C_1 e^{-2t} + C_2 t e^{-2t}}$

Next, VofP: $y_p = u_1 y_1 + u_2 y_2$, $y_1 = e^{-2t}$, $y_2 = te^{-2t}$

$u_1' = \dfrac{-y_2 g}{W}$, $u_2' = \dfrac{y_1 g}{W}$, $g = t^{-2}e^{-2t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}$$

$$= e^{-4t} - 2te^{-4t} - (-2te^{-4t}) = e^{-4t}$$

$\Rightarrow u_1' = -\dfrac{(te^{-2t})(t^{-2}e^{-2t})}{(e^{-4t})} = -t^{-1} \Rightarrow u_1 = -\ln|t|$

$\Rightarrow \underline{u_1 = -\ln t}$ since $t > 0$

and $u_2' = \dfrac{(e^{-2t})(t^{-2}e^{-2t})}{(e^{-4t})} = t^{-2} \Rightarrow u_2 = \int t^{-2}\,dt$

$\Rightarrow \underline{u_2 = -t^{-1}}$

$\Rightarrow y_p = (-\ln t)(e^{-2t}) + (-t^{-1})(te^{-2t})$

$= -(\ln t)e^{-2t} - \underbrace{e^{-2t}}_{\text{this is in } y_h} \Rightarrow \underline{y_p = -e^{-2t}\ln t}$

$\Rightarrow \underline{y = y_h + y_p = C_1 e^{-2t} + C_2 te^{-2t} - e^{-2t}\ln t}$

5. [20] Find the general solutions of the following differential equations:

(a) $y^{(4)} + 5y'' + 4y = 0$

(b) $y^{(4)} + 5y'' + 4y = 8t^2 + t$

(a) $y = e^{rt} \Rightarrow r^4 + 5r^2 + 4 = 0$

$\Rightarrow (r^2+4)(r^2+1) = 0 \Rightarrow r^2 = -4,\ r^2 = -1$

$\Rightarrow r = \pm i,\ \pm 2i$

$\Rightarrow \underline{y = C_1 \cos t + C_2 \sin t + C_3 \cos 2t + C_4 \sin 2t}$

(b) $y = y_h + y_p$, y_h was found in part (a).

MUC $\Rightarrow y_p = At^2 + Bt + C \Rightarrow y_p' = 2At + B$

$y_p'' = 2A \Rightarrow y_p''' = y_p^{(4)} = 0.$

Sub in: $[0] + 5[2A] + 4[At^2 + Bt + C] = 8t^2 + t$

$\Rightarrow (4A)t^2 + (4B)t + (10A + 4C)$

$= 8t^2 + t + 0$

Equate coefficients: $4A = 8 \Rightarrow A = 2$

$4B = 1 \Rightarrow B = 1/4$

$10A + 4C = 0 \Rightarrow 4C = -20 \Rightarrow C = -5$

$\Rightarrow \underline{y_p = 2t^2 + \frac{1}{4}t - 5}$

$\Rightarrow y = y_h + y_p$, where y_h was found in part (a).

6. [20] Use the Laplace transform to solve the initial value problem

$$y'' + 4y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = 0.$$

NO CREDIT will be awarded for any other method of solution.

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{e^{-2t}\}$ Let $Y(s) = \mathcal{L}\{y(t)\}$.

$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + 4[Y(s)] = \dfrac{1}{s-(-2)}$

#9 in table

#2 in table
$a = -2$

Use $y(0) = 1$, $y'(0) = 0$:

$$(s^2+4)Y(s) = \frac{1}{s+2} + s = \frac{1 + s(s+2)}{s+2}$$

$$\Rightarrow Y(s) = \frac{s^2 + 2s + 1}{(s+2)(s^2+4)}$$

Partial fractions: $Y(s) = \dfrac{A}{s+2} + \dfrac{Bs + C}{s^2+4}$ multiply by $(s+2)(s^2+4)$

$$\Rightarrow s^2 + 2s + 1 = A(s^2+4) + (Bs+C)(s+2)$$

$\underline{s=-2}$: $(-2)^2 + 2(-2) + 1 = A((-2)^2+4)$

$\Rightarrow 1 = 8A \Rightarrow \underline{A = 1/8}$

$\underline{s=0}$: $0^2 + 2(0) + 1 = (\frac{1}{8})(0^2+4) + (B(0)+C)(0+2)$

$\Rightarrow 1 = \frac{1}{2} + 2C \Rightarrow 2C = \frac{1}{2} \Rightarrow \underline{C = \frac{1}{4}}$

$\underline{s=-1}$: $(-1)^2 + 2(-1) + 1 = \frac{1}{8}((-1)^2+4) + (B(-1) + \frac{1}{4})(-1+2)$

$\Rightarrow 0 = \frac{5}{8} - B + \frac{1}{4} \Rightarrow \underline{B = \frac{5}{8} + \frac{1}{4} = \frac{7}{8}}$

$$\Rightarrow Y(s) = \frac{1/8}{s+2} + \frac{7}{8}\frac{s}{s^2+2^2} + \frac{1}{4}\left(\frac{1}{2}\right)\frac{2}{s^2+2^2}$$

$$\Rightarrow \underline{y(t) = \frac{1}{8}e^{-2t} + \frac{7}{8}\cos 2t + \frac{1}{8}\sin 2t}$$

table
#2 $a = -2$
#5 $a = 2$
#4 $a = 2$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
7.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$f'(t)$	$sF(s) - f(0)$
9.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
10.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
11.	$u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
12.	$e^{ct} f(t)$	$F(s-c)$