Math 204 Spring 2011
Exam III

Your printed name: Solutions

Your instructor's name: ____________________

Your section (or Class Meeting Days and Time): ____________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) for the duration of the exam.
3. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
4. Once the exam begins, you will have 60 minutes to complete your answers.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals, partial fraction decompositions, and matrix computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	20	20	20	20	20	100

1. [20] Solve the initial value problem

$$y'' + 4y = 1 - u_{3\pi}(t), \quad y(0) = 0, \quad y'(0) = 0.$$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{3\pi}(t)\}$. Let $Y(s) = \mathcal{L}\{y(t)\}$.

$\Rightarrow [s^2Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}] + 4[Y(s)] = \frac{1}{s} - \frac{e^{-3\pi s}}{s}$

$\Rightarrow (s^2+4)Y(s) = \frac{1}{s} - e^{-3\pi s}\frac{1}{s}$

(*) $\Rightarrow Y(s) = \frac{1}{s(s^2+4)} - e^{-3\pi s}\frac{1}{s(s^2+4)}$.

Let $H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$\Rightarrow 1 = A(s^2+4) + (Bs+C)(s)$

$\underline{s=0}$: $1 = 4A \Rightarrow \underline{A = 1/4}$

$\underline{s=2i}$: $1 = (2Bi + C)(2i) = -4B + 2Ci \Rightarrow \underline{B = -\frac{1}{4},\ C = 0}$

$\Rightarrow H(s) = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$

$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4} - \frac{1}{4}\cos 2t$

From (*) above: $Y(s) = H(s) - e^{-3\pi s}H(s)$

$\Rightarrow y(t) = h(t) - h(t-3\pi)u_{3\pi}(t)$

$\Rightarrow y(t) = \left[\frac{1}{4} - \frac{1}{4}\cos 2t\right] - \left[\frac{1}{4} - \frac{1}{4}\cos[2(t-3\pi)]\right]u_{3\pi}(t)$

2. [20] Solve the initial value problem

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -2.$$

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}$. Let $Y(s) = \mathcal{L}\{y\}$.

$$\Rightarrow [s^2 Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=-2}] - 2[sY(s) - \underbrace{y(0)}_{=1}] + 2[Y(s)] = e^{-2s}$$

$$\Rightarrow (s^2 - 2s + 2)Y(s) = s - 2 - 2 + e^{-2s} = s - 4 + e^{-2s}$$

$$\Rightarrow Y(s) = \frac{s-4}{s^2-2s+2} + e^{-2s}\frac{1}{s^2-2s+2}$$

$$s^2 - 2s + 2 = (s-1)^2 + 1$$

$$\Rightarrow Y(s) = \frac{s-4}{(s-1)^2+1} + e^{-2s}\frac{1}{(s-1)^2+1}$$

$$= \frac{(s-1)}{(s-1)^2+1^2} - 3\frac{1}{(s-1)^2+1^2} + e^{-2s}\frac{1}{(s-1)^2+1^2}$$

$$\Rightarrow \underline{y(t) = e^t \cos t - 3e^t \sin t + u_2(t)e^{t-2}\sin(t-2)}$$

3. [20] Solve the integral equation

$$y(t) + \int_0^t e^{-\tau} y(t-\tau)\, d\tau = 1.$$

$\mathcal{L}\{y\} + \mathcal{L}\{e^{-t} * y(t)\} = \mathcal{L}\{1\}$. Let $Y(s) = \mathcal{L}\{y\}$.

$\Rightarrow Y(s) + \mathcal{L}\{e^{-t}\}\mathcal{L}\{y\} = \frac{1}{s}$

$\Rightarrow Y(s) + \frac{1}{s+1} Y(s) = \frac{1}{s}$

$\Rightarrow \frac{s+1+1}{s+1} Y(s) = \frac{1}{s}$

$\Rightarrow Y(s) = \frac{s+1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

$\Rightarrow s+1 = A(s+2) + Bs$

$\underline{s=-2} \Rightarrow -1 = -2B \Rightarrow \underline{B = 1/2}$

$\underline{s=0} \Rightarrow 1 = 2A \Rightarrow \underline{A = 1/2}$

$\Rightarrow Y(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2}$

$\Rightarrow \underline{y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}}$

4. [20] Find the general solution of the differential equation system

$$x' = 3x + 6y,$$
$$y' = -x - 2y.$$

Show all of your work – calculators are NOT allowed for any matrix/vector operations.

$A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, $0 = \det(A - \lambda I) = \det\begin{bmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{bmatrix}$

$\Rightarrow 0 = (3-\lambda)(-2-\lambda) - (-1)(6) = -6 - \lambda + \lambda^2 + 6$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \underline{\lambda_1 = 0,\ \lambda_2 = 1}$

$(A - \underbrace{\lambda_1}_{=0} I)\vec{v} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 6 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 = R_1 + 3R_2} \left[\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then row 1 $\Rightarrow 3a + 6b = 0 \Rightarrow a = -2b$

$\Rightarrow \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ when $b = 1$.

$(A - \underbrace{\lambda_2}_{=1} I)\vec{v} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 2 & 6 & 0 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{R_2 = R_1 + 2R_2} \left[\begin{array}{cc|c} 2 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow$ row 1: $2a + 6b = 0 \Rightarrow a = -3b$

$\Rightarrow \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3b \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ when $b = 1$

$\Rightarrow$ gen soln is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

5. [20] Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Show all of your work – calculators are NOT allowed for any matrix/vector operations.

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}. \quad 0 = \det(A - \lambda I) = \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 4$$

$$\Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i, \quad \lambda_1 = 2i, \quad \lambda_2 = -2i$$

$$(A - \lambda_1 I)\vec{v} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \xrightarrow{R_2 = +iR_2} \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2i & +2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = R_1 - R_2} \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]. \quad \text{Let } \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}. \text{ Then row 1}$$

$$\Rightarrow -2ia + 2b = 0 \Rightarrow b = ai.$$

$$\text{Then } \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ ai \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i, \quad \text{when } a = 1.$$

$$\Rightarrow \vec{x} = c_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 2t\right)$$

$$+ c_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 2t\right) = \begin{bmatrix} c_1\cos 2t + c_2 \sin 2t \\ -c_1 \sin 2t + c_2 \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} c_1(1) + \cancel{c_2(0)} \\ \cancel{-c_1(0)} + c_2(1) \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = -3 \end{matrix}$$

$$\Rightarrow \vec{x} = 2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 2t\right) - 3\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 2t\right)$$

$$\text{or } \vec{x} = \begin{bmatrix} 2\cos 2t - 3\sin 2t \\ -3\cos 2t - 2\sin 2t \end{bmatrix}$$

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n, \ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
7.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$f'(t)$	$sF(s) - f(0)$
9.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
10.	$u_c(t)$	$\frac{e^{-cs}}{s}$
11.	$u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
12.	$e^{ct}f(t)$	$F(s-c)$
13.	$\delta(t-a)$	e^{-as}
14.	$\int_0^t f(t-\tau)\, g(\tau)\, d\tau$	$F(s)\, G(s)$