Sec. 2.2 Separable Equations

HW p.47: # 1, 11, 21, 30 Due: Friday, Sept. 3

Schaum's : pp. 21–29.

1^{st} order DEs have the form $y' = f(t,y)$

1^{st} order linear DEs have form $y' = -p(t)y + q(t)$ (or $a_1(t)y' + a_0(t)y = f(t)$)

1^{st} order separable DEs have form $y' = \underbrace{g(t)}_{\text{fcn. of } t \text{ alone}} \underbrace{h(y)}_{\text{fcn. of } y \text{ alone.}}$

To solve separable DEs we rewrite the eqn. as

$\frac{dy}{dt} = g(t)h(y)$ and separate variables : $\frac{dy}{h(y)} = g(t)dt$.

Then integrate : $\int \frac{dy}{h(y)} = \int g(t)dt$.

Ex 1 (Cf. #23, p.48) Find the general solution of

$$y' = 2y^2 + xy^2.$$

Soln: $y' = y^2(2+x)$ Separable!

$$\frac{dy}{y^2} = (2+x)dx$$

$$\int \frac{dy}{y^2} = \int (2+x)dx$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} + C$$

This equation defines y implicitly as a function of x. (Implicit Solution)

$$\boxed{\frac{-1}{\frac{x^2}{2} + 2x + C} = y}$$

(Explicit solution)

<u>Ex 2</u> (#22, p.48) Solve the IVP

$$y' = \frac{3x^2}{3y^2-4}, \quad y(1)=0$$

and determine the interval in which the solution is valid.

<u>Soln</u>: $\dfrac{dy}{dx} = 3x^2 \cdot \left(\dfrac{1}{3y^2-4}\right)$ (Variables are) separable!

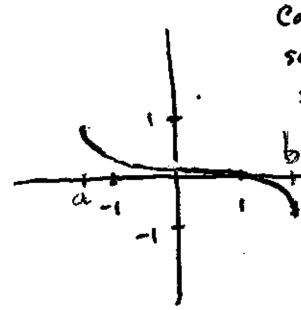
$$\int (3y^2-4)\,dy = \int 3x^2\,dx$$

$$y^3 - 4y = x^3 + C$$ ← This defines y implicitly as a function of x. It is extremely difficult (but not impossible!) to solve for y as an explicit function of x. We will proceed with the implicit form of the soln.

When $x=1$, we want $y=0$. Therefore

$$0^3 - 4(0) = 1^3 + C \Rightarrow C = -1$$

$$\therefore \boxed{y^3 - 4y = x^3 - 1}$$



Calculator gives soln. curve shown. Note vertical tangents at endpoints of (a,b).

In order to determine the largest interval $a<x<b$ containing $x=1$ in which the solution is valid, we examine those points where the derivative of y is undefined. Returning to the DE, we see that these occur when $3y^2-4=0 \Rightarrow y = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$.

If $y = \frac{2}{\sqrt{3}}$ then $\left(\frac{2}{\sqrt{3}}\right)^3 - 4\left(\frac{2}{\sqrt{3}}\right) = x^3 - 1 \Rightarrow \frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}} + 1 = x^3$

$$\Rightarrow \frac{8 - 24 + 3\sqrt{3}}{3\sqrt{3}} = x^3$$

$$\Rightarrow x = \sqrt[3]{\frac{3\sqrt{3}-16}{3\sqrt{3}}} \doteq -1.276337$$

If $y = -\frac{2}{\sqrt{3}}$ then a similar calculation gives $x = \sqrt[3]{\frac{3\sqrt{3}+16}{3\sqrt{3}}} \doteq 1.597810$

The solution is valid on $\boxed{\overbrace{-1.276337}^{a} < x < \overbrace{1.597810}^{b}}$ (approximately).

Ex 3 (Cf. #30, p. 49) Find the general solution to the DE

$$y' = \frac{y-x}{y+x}.$$

Soln: This DE is not separable since $\frac{y-x}{y+x} \neq g(x)h(y)$. Note, however, that the DE is equivalent to

$$(\dagger) \qquad y' = \frac{y/x - 1}{y/x + 1}.$$

The RHS depends on the ratio y/x. This suggests we make the change of variable $v = y/x$, or equivalently $y = vx$. Then $y' = v' \cdot x + 1 \cdot v$ so substituting in $(\dagger)$ yields

$$v'x + v = \frac{v-1}{v+1}$$

$$\Rightarrow \quad v' = \underbrace{\left(\frac{v-1}{v+1} - v\right)}_{g(v)} \underbrace{\frac{1}{x}}_{h(x)} \qquad \text{Separable!}$$

$$\frac{dv}{dx} = \left(\frac{\overbrace{v-1-v(v+1)}^{-1-v^2}}{v+1}\right) \cdot \frac{1}{x} \quad \Rightarrow \quad \frac{v+1}{v^2+1}\,dv = -\frac{1}{x}\,dx$$

$$\int \frac{v\,dv}{v^2+1} + \int \frac{1\,dv}{v^2+1} = -\int \frac{1}{x}\,dx \quad \Rightarrow \quad \frac{1}{2}\ln(v^2+1) + \text{Arctan}(v) = -\ln|x| + C$$

But $v = y/x$ so $\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \frac{1}{2}\ln(x^2) + \text{Arctan}(y/x) = C$

or $\boxed{\ln\sqrt{x^2+y^2} + \text{Arctan}(y/x) = C}$ (Implicit Form)

In polar coordinates, this is $\ln(r) + \theta = C$ or $\boxed{r = Ae^{-\theta}}$ $(A = e^C)$

(logarithmic spiral)