Sec. 2.4 Differences Between Linear and Nonlinear DEs

HW p.75: #1, 11, 23, 33 Due:

Schaum's: pp.73–81 (linear IVP's only)

Ex 1| Determine at least two solutions of the first order IVP.

$$ty' = 2y \,, \quad y(0) = 0 .$$

Solution: The DE is separable: $\frac{dy}{y} = \frac{2dt}{t} \Rightarrow \ln|y| = 2\ln|t| + c$. so $y = kt^2 \;(k = \pm e^c)$. $0 = y(0) = k(0)^2$ is automatically satisfied for any constant so $y(t) = kt^2$ solves the IVP for any choice of k.

Two solutions: $\boxed{\begin{array}{l} y_1(t) = t^2 \\ y_2(t) = 0 \end{array}}$ $\begin{array}{l} (k=1) \\ (k=0) \end{array}$

Ex 2| Explain why there is no solution of the first order IVP

$$(y')^2 + y^2 = 0 \,, \quad y(0) = 1 .$$

Solution: Suppose $y = y(t)$ solves the DE in an open interval I containing $t=0$. Then

$$[y'(t)]^2 + [y(t)]^2 = 0 \quad \text{for all } t \text{ in } I.$$

But the only way that two nonnegative expressions can have a sum of zero is if both expressions are zero. Therefore $y'(t) = 0 = y(t)$ for all t in I. But then $0 = y(0) \neq 1$.

Therefore there can be no solution to the IVP given above.

Q: When is a first order IVP

$$(9) \qquad y' = f(t,y) \ , \quad y(t_0) = y_0$$

guaranteed to have one and only one solution?

A: See the general Existence - Uniqueness Theorem (2.4.2) on p. 70.

Ask students to read theorem with you. No need to write statement on board.

Theorem 2.4.2

If the functions f and $\frac{\partial f}{\partial y}$ are continuous in some open rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then in some nonempty interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \varphi(t)$ of the initial value problem

$$(9) \qquad y' = f(t,y), \quad y(t_0) = y_0 .$$

Discuss structure of mathematical theorems as "conditional truth":

If [hypothesis], then [conclusion].

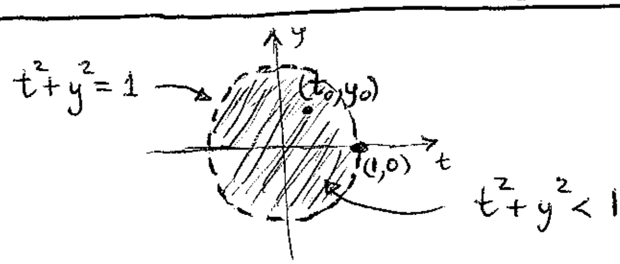
Ex 3 (Similar to #8, p. 75) State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied for the IVP

$$y' = (1 - t^2 - y^2)^{1/2}, \ y(t_0) = y_0 .$$

Solution: $f(t,y) = \left[1 - (t^2 + y^2)\right]^{1/2}$ is continuous if $t^2 + y^2 \le 1$.

$$\frac{\partial f}{\partial y} = \frac{1}{2}\left[1 - (t^2+y^2)\right]^{-1/2}(-2y) = \frac{-y}{\sqrt{1-(t^2+y^2)}} \quad \text{is continuous if } t^2 + y^2 < 1.$$

Therefore $\boxed{\text{the hypotheses of Theorem 2.4.2 are satisfied as long as } t_0^2 + y_0^2 < 1.}$



Note that we get no information about the size of the interval in which there exists a unique solution to the possibly <u>nonlinear</u> IVP

(9) $$y' = f(t,y), \quad y(t_0) = y_0 .$$

(See Theorem 2.4.2 on p. 70.) For <u>linear</u> IVP's

(1)-(2) $$y' + p(t)y = g(t), \quad y(t_0) = y_0,$$

the situation is different. We are guaranteed that there exists a unique solution to (1)-(2) on an interval of <u>fixed size</u> that depends on the coefficients p and g. See the linear Existence-Uniqueness Theorem (2.4.1 on p. 68).

Don't write statement of theorem 2.4.1 on the board. Ask students to read along from book.

Theorem 2.4.1
If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \varphi(t)$ that satisfies the DE

(1) $$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the I.C.

(2) $$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

<u>Ex 4</u> Determine an interval in which the solution to the linear IVP

(#2, p. 75) $$t(t-4)y' + y = 0, \quad y(2) = 1,$$

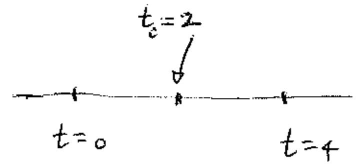
is certain to exist.

<u>Solution</u>: In order to apply the linear Existence-Uniqueness Theorem (2.4.1 on p. 68) we must place the DE in standard form. This is done by dividing the given DE through by $t(t-4)$:

$$(*) \qquad y' + \frac{1}{t(t-4)}y = 0 \ , \qquad y(\underset{t_0=2}{2}) = \overset{y_0=1}{1} .$$

The functions

$$p(t) = \frac{1}{t(t-4)} \qquad (t \neq 0, t \neq 4)$$



and

$$g(t) = 0 \qquad (t \text{ any real number})$$

are continuous on the open interval $I = (0,4)$ containing the point $t_0 = 2$. By Theorem 2.4.1 there exists a unique solution $y = y(t)$ to the IVP $(*)$ above on the interval $\boxed{I = (0,4)}$.