Chap. 3 Second Order Linear DEs

Sec. 3.1 Homogeneous Equations with Constant Coefficients

HW p. 144: #1, 12, 21, 28 Due: Wed., Sept. 15

Schaum's: pp. 83-88

In this chapter we will learn how to find the general solution of some second order linear DEs:

$$(*) \qquad a_2(t)y'' + a_1(t)y' + a_0(t)y = G(t).$$

If $G(t) \equiv 0$ then $(*)$ is called homogeneous; otherwise it is called nonhomogeneous. We first take up the important special of case of homogeneous linear eqns. with constant coefficients:

$$(**) \qquad ay'' + by' + cy = 0.$$

The key idea in solving $(**)$ is to look for solutions of exponential form: $y(t) = e^{rt}$ where r is a constant. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$ so substituting in $(**)$ yields

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2 + br + c) = 0$$

$$(\dagger) \qquad ar^2 + br + c = 0$$

The algebraic equation $(\dagger)$ is called the characteristic equation of the DE $(**)$.

Superposition Principle (Thm 3.2.2, p.147)

Ex 1 (#2, p.144) Solve the DE $\quad y'' + 3y' + 2y = 0$.

Solution: $y = e^{rt}$ leads to $r^2 + 3r + 2 = 0$ or $(r+2)(r+1) = 0$ so $r = -2$ or $r = -1$. That is, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$ are solutions of the DE. We anticipate the Superposition Principle for linear homogeneous equations (See Thm 3.2.2, p.147). It guarantees that

$$y(t) = c_1 e^{-2t} + c_2 e^{-t}$$

is a solution of the DE for any choice of constants c_1 and c_2. It is easy to verify this:

$$\begin{aligned} y'' + 3y'' + 2y &= (c_1e^{-2t} + c_2e^{-t})'' + 3(c_1e^{-2t} + c_2e^{-t})' + 2(c_1e^{-2t} + c_2e^{-t}) \\ &= \cancel{4c_1e^{-2t}} + \cancel{c_2e^{-t}} - \cancel{6c_1e^{-2t}} - \cancel{3c_2e^{-t}} + \cancel{2c_1e^{-2t}} + \cancel{2c_2e^{-t}} \\ &\overset{\checkmark}{=} 0 \end{aligned}$$

We will see later that $y(t) = c_1e^{-2t} + c_2e^{-t}$ is the most <u>general solution</u> possible for the DE. (See Thm. 3.2.4, p.149)

Ex 2 (#15, p.144) Find the solution of the IVP

$$y'' + 8y' - 9y = 0, \quad y(1) = 1,\ y'(1) = 0.$$

Sketch the graph of the solution and describe its behavior as $t \to \infty$.

Solution: $y = e^{rt}$ in $y'' + 8y' - 9y = 0$ leads to $r^2 + 8r - 9 = 0 \Rightarrow$ $(r+9)(r-1) = 0 \quad \Rightarrow \quad r = -9$ or $r = 1$. Then $y_1(t) = e^{-9t}$ and $y_2(t) = e^t$ solve the DE. By the superposition principle,

$$y = c_1 e^{-9t} + c_2 e^t$$

solves the DE for arbitrary constants c_1 and c_2. We will see later (Theorem 3.2.4, Sec. 3.2) that this is the most general solution possible for the DE.

We need to choose c_1 and c_2 so the initial conditions are satisfied.
Note that $y'(t) = -9c_1e^{-9t} + c_2e^{t}$. Therefore

$$1 = y(1) = c_1e^{-9} + c_2e \quad \text{and} \quad 0 = y'(1) = -9c_1e^{-9} + c_2e$$

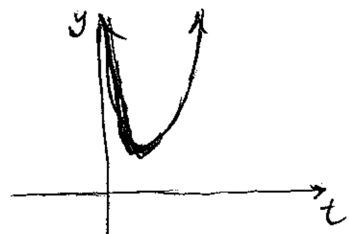
We subtract these two equations to eliminate c_2:

$$1 - 0 = c_1e^{-9} + \cancel{c_2e} - (-9)c_1e^{-9} - \cancel{c_2e} = 10c_1e^{-9} \Rightarrow c_1 = \frac{e^{9}}{10}.$$

Substituting in the second equation yields $0 = -9\left(\frac{e^9}{10}\right)e^{-9} + c_2e \Rightarrow c_2 = \frac{9}{10}e^{-1}$.

Consequently, $y(t) = \frac{e^9}{10}e^{-9t} + \frac{9}{10}e^{-1}e^{t} = \boxed{\frac{1}{10}e^{-9(t-1)} + \frac{9}{10}e^{t-1}}$ solves the IVP.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{1}{10}e^{-9(t-1)} + \frac{9}{10}e^{t-1}\right) = +\infty.$$



Ex 31 (#24, p.144) Determine the values of α, if any, for which <u>all</u> solutions of

$$y'' + (3-\alpha)y' - 2(\alpha-1)y = 0$$

tend to zero as $t \to \infty$. Also, determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

<u>Solution:</u> $y = e^{rt}$ in the DE leads to $r^2 + (3-\alpha)r - 2(\alpha-1) = 0$ or $(r+2)(r-(\alpha-1)) = 0$ so $r = -2$ or $r = \alpha - 1$. Then the general solution is $y(t) = c_1e^{-2t} + c_2e^{(\alpha-1)t}$ (at least, if $\alpha \neq -1$). Both terms will go to zero as $t \to \infty$ if $\boxed{\alpha < 1}$. Clearly, there is $\boxed{\text{no value of } \alpha}$ for which all (nonzero) solutions become unbounded as $t \to \infty$, since $c_1e^{-2t} \to 0$ as $t \to \infty$.