Sec. 3.2 Solutions of Linear Homogeneous Equations; the Wronskian

HW p.155: # 7, 17, 24, 31 Due: Fri., Sept. 17

Schaum's: pp. 73–82

Q: When is a linear second order IVP

(4) $y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$

guaranteed to have exactly one solution?

A: Theorem 3.2.1, p. 146 (Existence and Uniqueness Theorem)

Don't write on the board. Ask students to read along on p. 146.

Consider the linear second order IVP (4) where p, q, and g are continuous functions on an open interval I that contains the point t_0. Then there is exactly one solution $y = \varphi(t)$ of this IVP and the solution exists throughout the interval I.

Ex 1 (#8, p. 155) Determine the largest interval in which the IVP

$$(t-1)y'' - 3ty' + 4y = \sin(t), \quad y(-2) = 2, \quad y'(-2) = 1$$

is certain to have a unique (twice differentiable) solution.

Solution: In order to make use of the existence/uniqueness theorem 3.2.1 we must place the DE in standard form:

$$y'' + p(t)y' + q(t)y = g(t).$$

Our DE is equivalent to

$$y'' - \frac{3t}{t-1}y' + \frac{4}{t-1}y = \frac{\sin(t)}{t-1}.$$

$$\left.\begin{aligned} p(t) &= \frac{-3t}{t-1} \\ q(t) &= \frac{4}{t-1} \\ g(t) &= \frac{\sin(t)}{t-1} \end{aligned}\right\} \text{ are continuous as long as } t \neq 1.$$

Since $t_0 = -2 \in (-\infty, 1)$, the largest interval in which the given IVP is certain to have exactly one solution is

$$\boxed{I = (-\infty, 1)},$$

according to Theorem 3.2.1

Q2: What is the form of the most general solution of the linear, homogeneous, second order DE

(2) $$y'' + p(t)y' + q(t)y = 0 \quad ?$$

In order to answer this question we need two concepts.

① Superposition Principle (Theorem 3.2.2, p.147)

Don't write on board. Read together, p.147.

If y_1 and y_2 are two solutions of the DE (2), then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

② Def: The Wronskian of two differentiable functions f and g is the function

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

$$= f(t)g'(t) - f'(t)g(t).$$

Ex 2 (#3, p. 155) Find the Wronskian of the pair of functions $f(t) = e^{-2t}$, $g(t) = te^{-2t}$.

Solution: $W(f,g)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ (e^{-2t})' & (te^{-2t})' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}-2te^{-2t} \end{vmatrix} =$

$$e^{-2t} \cdot e^{-2t}(1-2t) - (-2e^{-2t})(te^{-2t}) = e^{-4t}(1-2t) + 2te^{-4t} = \boxed{e^{-4t}}.$$

Omit this example if pressed for time.

Ex 3 (#18, p. 155) If the Wronskian of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.

Solution: $t^2e^t = W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t)$.

Therefore $y = g(t)$ is a solution to $ty' - y = t^2e^t$. This is

(*) $$y' - \frac{1}{t}y = te^t$$

in standard form (linear, first order DE). An integrating factor is

$$e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln(t) + \cancel{c}^{\,0}} = e^{\ln(t^{-1})} = t^{-1}.$$

Multiplying through by t^{-1} gives

$$\underbrace{t^{-1}y' - t^{-2}y}_{\text{Exact}} = e^t$$

$$\frac{d}{dt}\{t^{-1}y\} = e^t$$

Integrating both sides gives

$$t^{-1}y = \int e^t dt = e^t + c$$

$$y = te^t + ct.$$

Therefore $\boxed{g(t) = te^t + ct}$ where c is an arbitrary constant.

A2: The most general solution on an interval I to

(2) $$y'' + p(t)y' + q(t)y = 0$$

is $y(t) = c_1y_1(t) + c_2y_2(t)$ where y_1 and y_2 are two solutions of (2) on I for which $W(y_1,y_2)(t_0) \neq 0$ for some point $t_0 \in I$ and where c_1, c_2 are arbitrary constants [Theorem 3.2.4, p. 149].

Note: Such a pair of solutions $y = y_1(t)$ and $y = y_2(t)$ of (2) is called a fundamental set of solutions (F.S.S.)

$$y_2' = xe^x + 1\cdot e^x = (x+1)e^x$$
$$y_2'' = (x+1)\cdot e^x + 1\cdot e^x = (x+2)e^x$$

Ex 4 (#26, p. 156) Verify that $y_1(x) = x$ and $y_2(x) = xe^x$ are solutions of

$$x^2y'' - x(x+2)y' + (x+2)y = 0 \quad \text{on the interval } x>0.$$

Do they constitute a fundamental set of solutions?

Solution:

$$x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = x^2\cdot 0 - x(x+2)\cdot 1 + (x+2)x \overset{\checkmark}{=} 0$$

$$x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(x+2)e^x - x\underbrace{(x+2)(x+1)}_{x^2+3x+2}e^x + (x+2)xe^x$$
$$= \left[\cancel{x^3} + \cancel{2x^2} - \cancel{x^3} - \cancel{3x^2} - \cancel{2x} + \cancel{x^2} + \cancel{2x}\right]e^x$$
$$\overset{\checkmark}{=} 0$$

$$W(y_1,y_2)(x) = \begin{vmatrix} x & xe^x \\ 1 & (x+1)e^x \end{vmatrix} = x(x+1)e^x - xe^x = x^2e^x \neq 0 \text{ on } x>0.$$

$\boxed{\text{Yes,}}$ $y_1(x) = x$ and $y_2(x) = xe^x$ form a F.S.S. of the DE on $x>0$.

The general solution is $y(x) = c_1x + c_2xe^x$ where c_1, c_2 are arbitrary constants.

Ex 5 (#32, p. 156) Find the Wronskian of two solutions of

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \qquad \text{(Legendre's DE)}$$

without solving the equation.

Solution: Let y_1 and y_2 be two solutions of the DE. We place the DE in standard form

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

and use Abel's Theorem [3.2.6 on p. 153]:

$$W(y_1,y_2)(x) = c\exp\left[-\int p(x)dx\right] = c\exp\left[\int \frac{2x}{1-x^2}dx\right] = c\exp\left(-\ln|1-x^2|\right) = \frac{c}{|1-x^2|},$$

where c is ~~certain~~ certain constant.