Sec. 3.3 Complex Roots of the Characteristic Equation

HW p. 163: # 7, 17, 25, 35 Due: Mon., Sept. 20

Schaum's: pp. 83 – 88

Consider (*): $ay'' + by' + cy = 0$ where a, b, c are real constants with $a \neq 0$.

Then $y = e^{rt}$ leads to (†): $ar^2 + br + c = 0$ the characteristic eqn. of (*).

	Roots of (†)	F.S.S. of (*)	Gen. Soln. of (*)	
$b^2 - 4ac > 0$	Real, distinct: $r = r_1, r = r_2$	$e^{r_1 t}, e^{r_2 t}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$	← Sec. 3.1
$b^2 - 4ac < 0$	Complex conjugates $r = \lambda \pm i\mu$	$e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)$	$y = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$	← Sec. 3.3
$b^2 - 4ac = 0$	Real, repeated $r = r_1$ (mult. two)			← Sec. 3.4

Ex 1 (Similar to #12, p. 163) Find the general solution of $4y'' + y = 0$.

Soln: $y = e^{rt}$ leads to $4r^2 + 1 = 0 \Rightarrow r^2 = -\frac{1}{4} \Rightarrow r = \pm \frac{i}{2}$.

We are thus led to solutions of the DE of the form $y_1 = e^{\frac{it}{2}}$ and $y_2 = e^{-\frac{it}{2}}$.

What do these formulas mean?

Digression on Power Series:

Recall the following Maclaurin (power) series representations from Calculus II:

① $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

② $$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

③ $$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

These power series converge to the function in the left member of each respective equation for all real numbers z, as you learned in Calculus II. In fact, these power series converge for every <u>complex</u> number $z = x + iy$ and are used to define the complex exponential, cosine, and sine functions. We take $z = i\theta$ in equation ①, where θ is an arbitrary real number:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

Using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc. we have

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

Comparing this identity with equations ② and ③ above yields <u>Euler's ormula</u>:

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

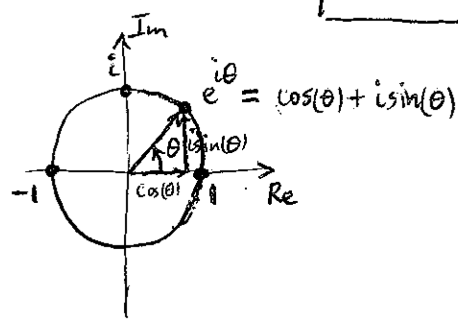
For instance, taking $\theta = \pi/2$ in Euler's formula gives

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = 0 + 1i \quad \text{so} \quad \boxed{e^{i\pi/2} = i}.$$

Likewise

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0\cdot i \quad \text{so} \quad \boxed{e^{i\pi} = -1}.$$

Geometrical content of Euler's formula



Completion of Ex 1 Applying Euler's formula with $\theta = \pm \frac{t}{2}$ gives

$$y_1 = e^{it/2} = \cos\left(\tfrac{t}{2}\right) + i\sin\left(\tfrac{t}{2}\right)$$

$$y_2 = e^{-it/2} = \cos\left(\tfrac{-t}{2}\right) + i\sin\left(\tfrac{-t}{2}\right) = \cos(t) - i\sin\left(\tfrac{t}{2}\right)$$

as solutions to the DE $4y'' + y = 0$. It follows from the Superposition principle (Thm 3.2.2, p. 147) that

$$\left.\begin{array}{ll} & \tilde{y}_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = \cos(t/2) \\ \text{and} & \\ & \tilde{y}_2 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \sin(t/2) \end{array}\right\} \text{real valued!}$$

are also solutions to $4y'' + y = 0$. Since

$$W(\tilde{y}_1, \tilde{y}_2)(t) = \begin{vmatrix} \cos(t/2) & \sin(t/2) \\ -\frac{1}{2}\sin(t/2) & \frac{1}{2}\cos(t/2) \end{vmatrix} = \frac{1}{2} \neq 0$$

the pair $\tilde{y}_1 = \cos(t/2)$ and $\tilde{y}_2 = \sin(t/2)$ a F.S.S. to $4y'' + y = 0$ (on any interval). Therefore

$$\boxed{y = c_1\cos(t/2) + c_2\sin(t/2)}$$

is the general solution of $4y'' + y = 0$.

Ex 2 (#10, p. 163) Find the general solution of $y'' + 2y' + 2y = 0$.

Soln: $y = e^{rt}$ leads to $r^2 + 2r + 2 = 0$. Then applying the quadratic formula yields

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

We have (complex-valued) solutions

$$y_1 = e^{(-1+i)t} = e^{-t+it} = e^{-t} \cdot e^{it} = e^{-t}\left(\cos(t) + i\sin(t)\right) = e^{-t}\cos(t) + i e^{-t}\sin(t)$$

$$y_2 = e^{(-1-i)t} = e^{-t-it} = e^{-t} \cdot e^{-it} = e^{-t}\left(\cos(t) - i\sin(t)\right) = e^{-t}\cos(t) - i e^{-t}\sin(t)$$

By Superposition,

$$\tilde{y}_1 = \tfrac{1}{2}y_1 + \tfrac{1}{2}y_2 = e^{-t}\cos(t)$$

and

$$\tilde{y}_2 = \tfrac{1}{2i}y_1 - \tfrac{1}{2i}y_2 = e^{-t}\sin(t)$$

real-valued!

are also solutions to the DE. One checks that

$$W(\tilde{y}_1, \tilde{y}_2)(t) = \begin{vmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ -e^{-t}(\cos(t) + \sin(t)) & e^{-t}(\cos(t) - \sin(t)) \end{vmatrix} = e^{-2t} \neq 0$$

Therefore $\tilde{y}_1 = e^{-t}\cos(t)$, $\tilde{y}_2 = e^{-t}\sin(t)$ forms a F.S.S. of $y'' + 2y' + 2y = 0$ (on any interval). Thus

$$\boxed{y = e^{-t}\left(c_1\cos(t) + c_2\sin(t)\right)}$$ (c_1, c_2 arbitrary constants)

is the general solution of the DE.

(Fill in the second line in the table on p. 1 of these notes.)

similar to #35 & 38, p.165

<u>Ex 3</u> (#36, p.165) Use the substitution $x = \ln(t)$ in the Euler equation

$$(*) \qquad t^2 y'' + 4ty' + 2y = 0 \qquad (t > 0)$$

to convert it into an equivalent second order linear DE with constant coefficients. Use this to help solve the equation (*).

<u>Soln</u>: Let $x = \ln(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \qquad \text{so} \qquad t\frac{dy}{dt} = \frac{dy}{dx}.$$

Also

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(t^{-1}\frac{dy}{dx}\right) = -t^{-2}\frac{dy}{dx} + t^{-1}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= -t^{-2}\frac{dy}{dx} + t^{-1}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}$$

$$= -t^{-2}\frac{dy}{dx} + t^{-2}\frac{d^2y}{dx^2}$$

so $\quad t^2\dfrac{d^2y}{dt^2} = \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}$. Substituting from the circled identities in (*) gives

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4\frac{dy}{dx} + 2y = 0$$

$$(**) \qquad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \qquad \text{(Constant Coefficients!)}$$

$y = e^{rx}$ leads to $r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow r = -2, r = -1$.

The general solution of (**) is $y = c_1 e^{-2x} + c_2 e^{-x}$. Substituting $x = \ln(t)$ yields

$y = c_1 e^{-2\ln(t)} + c_2 e^{-\ln(t)} = c_1 e^{\ln(t^{-2})} + c_2 e^{\ln(t^{-1})} = \boxed{c_1 t^{-2} + c_2 t^{-1}}$ as the general solution of (*).