Sec. 3.4: Repeated Roots; Reduction of Order

HW p. 171 : #6, 11, 25, 41

Schaum's pp. 83–88

In Sec. 3.2 we learned that the general solution to

$$(*) \qquad y'' + p(t)y' + q(t)y = 0$$

is $y = c_1 y_1(t) + c_2 y_2(t)$ where $\{y_1(t), y_2(t)\}$ is a F.S.S. of $(*)$

Q: Suppose we knew one solution, say $y_1(t)$, to $(*)$. How can we find a second solution $y_2(t)$ to form a F.S.S?

A: (d'Alembert's method of reduction of order) Assume $y_2(t) = u(t)\,y_1(t)$ where $u(t)$ is a nonconstant function of t.

Ex 1 (Similar to #25 & 28, p. 173) Given that $y_1(t) = t$ is a solution to $t^2y'' - t(t+2)y' + (t+2)y = 0$ on $t > 0$, use the method of reduction of order to find a second solution.

Soln: We assume $y_2(t) = u(t)y_1(t) = tu$. Then $y_2' = tu' + 1u$ and $y_2'' = tu'' + 1\cdot u' + u' = tu'' + 2u'$. We want

$$t^2 y_2'' - t(t+2)y_2' + (t+2)y_2 = 0$$

So substituting from above the expressions for y_2, y_2', and y_2'' yields

$$t^2(tu'' + 2u') - \overbrace{t(t+2)}^{t^2+2t}(tu' + u) + (t+2)tu = 0$$

$$\Rightarrow \quad t^3u'' + \cancel{2t^2u'} - t^3u' - \cancel{t^2u} - \cancel{2t^2u'} - \cancel{2tu} + \cancel{t^2u} + \cancel{2tu} = 0$$

$$u'' - u' = 0 \qquad \text{Let } v = u'. \text{ Then } v' = u'' \text{ so}$$

[This is a constant coefficient, second order, linear DE but we will not use $u = e^{rt}$. Instead ...]

the previous equation becomes $v' - v = 0$

$$\frac{dv}{dt} = v$$

$$\ln|v| = \int \frac{dv}{v} = \int dt = t + c$$

$$\Rightarrow \quad v(t) = Ae^{t} \qquad (A = \pm e^{c})$$

But $u' = v$ so $u = \int Ae^{t}\,dt = Ae^{t} + B$

$$\therefore \quad y_2 = uy_1 = (Ae^{t} + B)t = Ate^{t} + Bt$$

We already knew that $y_1(t) = t$ is a solution. Taking $A = 1$ and $B = 0$ we get a second solution: $\boxed{y_2(t) = te^{t}}$

<u>Ex 2</u> (#4, p.171) Find the general solution of $4y'' + 12y' + 9y = 0$.

<u>Soln:</u> $y = e^{rt}$ leads to $4r^2 + 12r + 9 = 0$ so $(2r+3)(2r+3) = 0$

$\Rightarrow r = -\frac{3}{2}$ (multiplicity two). Therefore $y_1(t) = e^{-\frac{3}{2}t}$ is a solution.

We use the method of reduction of order to find a second solution.

Assume $y_2 = uy_1 = ue^{-\frac{3}{2}t}$. Then $y_2' = u'e^{-\frac{3}{2}t} - \frac{3}{2}ue^{-\frac{3}{2}t}$

and $y_2'' = u''e^{-\frac{3}{2}t} - \frac{3}{2}u'e^{-\frac{3}{2}t} - \frac{3}{2}u'e^{-\frac{3}{2}t} + \frac{9}{4}ue^{-\frac{3}{2}t}$

$$= \left(u'' - 3u' + \frac{9}{4}u\right)e^{-\frac{3}{2}t}$$

We want $4y_2'' + 12y_2' + 9y_2 = 0$

so
$$4\left(u'' - \cancel{3u'} + \cancel{\tfrac{9}{4}u}\right)e^{-\frac{3}{2}t} + 12\left(\cancel{u'} - \cancel{\tfrac{3}{2}u}\right)e^{-\frac{3}{2}t} + \cancel{9ue^{-\frac{3}{2}t}} = 0$$

$$\Rightarrow \quad 4u'' = 0 \quad \Rightarrow \quad u = c_1 t + c_2$$

Therefore $y_2 = u e^{-\frac{3}{2}t} = (c_1 t + c_2) e^{-\frac{3}{2}t} = c_1 t e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}$

We knew that $y_1(t) = e^{-\frac{3}{2}t}$ is a solution. We can get a second solution by setting $c_2 = 0$ and $c_1 = 1$: $y_2(t) = t e^{-\frac{3}{2}t}$.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-\frac{3}{2}t} & t e^{-\frac{3}{2}t} \\ -\frac{3}{2} e^{-\frac{3}{2}t} & (1 - \frac{3}{2}t) e^{-\frac{3}{2}t} \end{vmatrix} = (1 - \cancel{\tfrac{3}{2}t}) e^{-3t} + \cancel{\tfrac{3}{2}t} e^{-3t} = e^{-3t} \neq 0$$

Therefore $y_1(t) = e^{-\frac{3}{2}t}$, $y_2(t) = t e^{-\frac{3}{2}t}$ forms a F.S.S. The general solution is $\boxed{y(t) = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t}}$.

Summary: To solve $(*)$ $ay'' + by' + cy = 0$, $y = e^{rt}$ leads to $(\dagger)$ $ar^2 + br + c = 0$.

	Roots of $(\dagger)$	F.S.S. of $(*)$	Gen. Soln. of $(*)$
$b^2 - 4ac > 0$	Real, distinct: $r = r_1, r = r_2$	$e^{r_1 t}$, $e^{r_2 t}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$b^2 - 4ac < 0$	Complex conjugates: $r = \lambda \pm i\mu$	$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$	$y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$
$b^2 - 4ac = 0$	Real, repeated: $r = r_1 = r_2$	$e^{r_1 t}$, $t e^{r_1 t}$	$y = e^{r_1 t}(c_1 + c_2 t)$

(#42, p.174)

Ex 31 Use the substitution $x = \ln(t)$ introduced in #34 in Sec. 3.3 to solve the Euler equation

$$t^2 y'' + 2ty' + \frac{1}{4}y = 0$$

on the interval $t > 0$.