Sec. 3.6 Variation of Parameters

HW p.189: # 5, 10, 13   Due: Mon., Sept. 26

Schaum's: pp. 103–109

In this section we develop a general method (variation of parameters) for finding a particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

provided we can find a fundamental set of solutions $y_1(t)$, $y_2(t)$ to the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$.

Ex 1 (#8, p.189) Find the general solution of $y'' + 4y = \boxed{3\csc(2t)}$ (†) on the interval $0 < t < \pi/2$.

Solution: First, we solve the associated homogeneous equation

(*) $$y'' + 4y = 0,$$

$y = e^{rt}$ in (*) leads to $r^2 + 4 = 0$ so $r = \pm 2i$. The solution in the complex roots case $r = \lambda \pm i\mu$ is

$$y = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t)).$$

Therefore $y_c(t) = c_1\underbrace{\cos(2t)}_{y_1(t)} + c_2\underbrace{\sin(2t)}_{y_2(t)}$. Next, we need a particular solution of the nonhomogeneous equation (†). We introduce the method of variation of parameters. We assume

Discuss why MoUC won't work!

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$= u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

where $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are parameters (functions) to be determined so that y_p solves $(\dagger)$. Note that

$$y_p' = u_1'\cos(2t) - 2u_1\sin(2t) + u_2'\sin(2t) + 2u_2\cos(2t)$$

and

$$y_p'' = \frac{d}{dt}\left[u_1'\cos(2t) + u_2'\sin(2t)\right] - 2u_1'\sin(2t) - 4u_1\cos(2t) + 2u_2'\cos(2t) - 4u_2\sin(2t)$$

We want

$$y_p'' + 4y_p \overset{\text{Want}}{=} 3\csc(2t)$$

$$\frac{d}{dt}\left[u_1'\cos(2t) + u_2'\sin(2t)\right] - 2u_1'\sin(2t) + 2u_2'\cos(2t) - 4\left[\cancel{u_1\cos(2t) + u_2\sin(2t)}\right]$$
$$+ 4\left[\cancel{u_1\cos(2t) + u_2\sin(2t)}\right] = 3\csc(2t)$$

$$\frac{d}{dt}\left[u_1'\cos(2t) + u_2'\sin(2t)\right] - 2u_1'\sin(2t) + 2u_2'\cos(2t) = 3\csc(2t)$$

We see that a solution results if we set

$$(**)\quad \begin{cases} u_1'\cos(2t) + u_2'\sin(2t) = 0 \\ \text{and} \\ -2u_1'\sin(2t) + 2u_2'\cos(2t) = 3\csc(2t) \end{cases}$$

Thus we have a system of two (linear) equations in the 2 unknowns u_1' and u_2'.

<u>Recall Cramer's Rule</u>: The solution of the system $\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$

is given by $x = \dfrac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

Applying this to $(**)$ yields

$$u_1' = \frac{\begin{vmatrix} 0 & \sin(2t) \\ 3\csc(2t) & 2\cos(2t) \end{vmatrix}}{\begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}} = \frac{-3\csc(2t)\sin(2t)}{2\cos^2(2t) + 2\sin^2(2t)} = \frac{-3}{2} \Rightarrow u_1(t) = -\frac{3}{2}t + \overset{0}{\cancel{c}}$$

Wronskian of $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$

$$u_2' = \frac{\begin{vmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 3\csc(2t) \end{vmatrix}}{\begin{vmatrix} & \\ & \end{vmatrix}} = \frac{3\cos(2t)\csc(2t)}{2} \Rightarrow u_2 = \frac{3}{2}\int \frac{\cos(2t)}{\sin(2t)}\,dt = \frac{3}{4}\ln|\sin(2t)| + \overset{0}{\cancel{c}}$$

Therefore $y_p = u_1\cos(2t) + u_2\sin(2t) = -\frac{3}{2}t\cos(2t) + \frac{3}{4}\sin(2t)\ln|\sin(2t)|$

We conclude by writing the general solution:

$$y = y_c + y_p$$

$$\boxed{y = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{2}t\cos(2t) + \frac{3}{4}\sin(2t)\ln|\sin(2t)|}$$

This process can be applied to find a particular solution for

$$y'' + p(t)y' + q(t)y = g(t)$$

provided $y_1(t), y_2(t)$ forms a F.S.S. to the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$. It yields

$$\boxed{\begin{gathered} y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \\ \text{where} \quad u_1(t) = \int \frac{-g(t)y_2(t)}{W(y_1,y_2)(t)}\,dt \quad \text{and} \quad u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)(t)}\,dt. \end{gathered}}$$

(See the textbook pp. 187–188.)

Ex 2 (#18, p.189) Verify that $y_1(x) = x^{-1/2}\sin(x)$, $y_2(x) = x^{-1/2}\cos(x)$ form a fundamental set of solutions to $x^2y'' + xy' + (x^2 - 1/4)y = 0$ on the interval $x > 0$. Then find the general solution to

$$x^2y'' + xy' + (x^2 - 1/4)y = 3x^{3/2}\sin(x) \qquad \text{on} \quad x > 0.$$

Solution: If $x>0$ and $y_1 = x^{-1/2}\sin(x)$ then $y_1' = -\frac{1}{2}x^{-3/2}\sin(x) + x^{-1/2}\cos(x)$

$$\text{and } y_1'' = \frac{3}{4}x^{-5/2}\sin(x) - \frac{1}{2}x^{-3/2}\cos(x) - \frac{1}{2}x^{-3/2}\cos(x) - x^{-1/2}\sin(x)$$

$$= \frac{3}{4}x^{-5/2}\sin(x) - x^{-3/2}\cos(x) - x^{-1/2}\sin(x).$$

Therefore
$$x^2y_1'' + xy_1' + (x^2 - \tfrac{1}{4})y_1 = \cancel{\tfrac{3}{4}x^{-1/2}\sin(x)} - \cancel{x^{1/2}\cos(x)} - \cancel{x^{3/2}\sin(x)} - \cancel{\tfrac{1}{2}x^{-1/2}\sin(x)} + \cancel{x^{1/2}\cos(x)}$$
$$+ \cancel{x^{3/2}\sin(x)} - \cancel{\tfrac{1}{4}x^{-1/2}\sin(x)}$$

$$= 0.$$

A completely analogous (routine) calculation shows $x^2y_2'' + xy_2' + (x^2 - \frac{1}{4})y_2 = 0$ for $x > 0$. That is y_1 and y_2 solve the homogeneous DE on $x > 0$.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^{-1/2}\sin(x) & x^{-1/2}\cos(x) \\ -\frac{1}{2}x^{-3/2}\sin(x) + x^{-1/2}\cos(x) & -\frac{1}{2}x^{-3/2}\cos(x) - x^{-1/2}\sin(x) \end{vmatrix}$$

$$= \cancel{-\tfrac{1}{2}x^{-2}\sin(x)\cos(x)} - x^{-1}\sin^2(x) + \cancel{\tfrac{1}{2}x^{-2}\sin(x)\cos(x)} - x^{-1}\cos^2(x)$$

$$= \frac{-1}{x}$$

So $W(y_1, y_2)(x) \neq 0$ for $x > 0$. Therefore y_1, y_2 form a F.S.S. on $x > 0$.

In order to use the variation of parameters formula for a particular solution, we must place the DE in standard form by dividing through by x^2:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \underbrace{3x^{-1/2}\sin(x)}_{g(x)}.$$

Therefore $y_p(x) = u_1(x)\overbrace{x^{-1/2}\sin(x)}^{y_1(x)} + u_2(x)\overbrace{x^{-1/2}\cos(x)}^{y_2(x)}$ where

$$u_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1,y_2)(x)}dx = \int \frac{-3\cancel{x^{-1/2}}\sin(x)\cancel{x^{-1/2}}\cos(x)}{-\cancel{1/x}}dx = 3\int \overbrace{\sin(x)}^{u}\overbrace{\cos(x)dx}^{du}$$

$$= \frac{3}{2}\sin^2(x) + \cancelto{0}{c}$$

$$u_2(x) = \int \frac{g(x)y_1(x)}{W(y_1,y_2)(x)}dx = \int \frac{3\cancel{x^{-1/2}}\sin(x)\cancel{x^{-1/2}}\sin(x)}{-\cancel{1/x}}dx = -3\int \sin^2(x)dx$$

Recall that $\cos(2\theta) = 1 - 2\sin^2(\theta)$ so $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$.

$$\therefore u_2(x) = -3\int\left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]dx = -\frac{3}{2}x + \underbrace{\frac{3}{4}\sin(2x)}_{\frac{3}{2}\sin(x)\cos(x)} + \cancelto{0}{c}$$

$$\therefore y_p(x) = u_1y_1 + u_2y_2 = \frac{3}{2}\sin^2(x)x^{-1/2}\sin(x) + \left[-\frac{3x}{2} + \frac{3}{4}\sin(2x)\right]x^{-1/2}\cos(x)$$

$$= \frac{3}{2}x^{-1/2}\sin^3(x) - \frac{3}{2}x^{1/2}\cos(x) + \frac{3}{2}x^{-1/2}\sin(x)\cos^2(x)$$

$$= \frac{3}{2}x^{-1/2}\sin(x)\left[\sin^2(x) + \cos^2(x)\right] - \frac{3}{2}x^{1/2}\cos(x)$$

$$= \frac{3}{2}x^{-1/2}\sin(x) - \frac{3}{2}x^{1/2}\cos(x)$$

Gen. Soln:

$$y = y_c + y_p = \overbrace{c_1x^{-1/2}\sin(x) + c_2x^{-1/2}\cos(x) + \frac{3}{2}x^{-1/2}\sin(x)}^{\text{combine}} - \frac{3}{2}x^{1/2}\cos(x)$$

$$\boxed{y = \tilde{c}_1x^{-1/2}\sin(x) + c_2x^{-1/2}\cos(x) - \frac{3}{2}x^{1/2}\cos(x)}$$