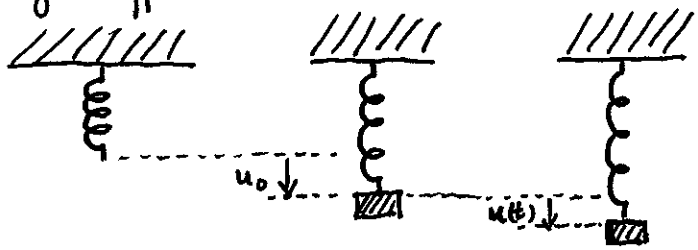
Sec. 3.7: Mechanical and Electrical Vibrations

HW p. 202: #7, 9, 12, 27   Due: Wed., Sept. 28

Schaum's: pp. 114 – 130.

Consider a body of mass m attached to a vertical spring hanging from a rigid support.



System in Static Equilibrium   Snapshot of System in Dynamic Equilibrium

Goal: Find the equation governing the vertical displacement $u(t)$ of the body from its static equilibrium position at time t.

Assumptions:

① All motion of the body lies in a plane (represented by the blackboard) and is strictly up and down in that plane. (In particular, there is no twisting or side-to-side swinging.)

Convention: We will take downward displacements, velocities, and forces to be positive.

② The body's motion is governed by Newton's 2^{nd} law:

$$m\vec{a} = \vec{F}_{net}$$

③ The only forces that act on the body are gravity, the spring force, the damping (or resistive) force exerted by the medium as the body moves through it, and a time-varying external force:

$$\vec{F}_{net} = \vec{F}_{gravity} + \vec{F}_{spring} + \vec{F}_{damping} + \vec{F}_{external}$$

④ $\vec{F}_{gravity} = mg$ where g is the (constant) acceleration of gravity near the earth's surface

⑤ $\vec{F}_{spring} = -k(u_0 + u)$ Hooke's Law: The force exerted by the spring is proportional to the stretch (or compression) of the spring and acts in the opposite direction from the stretch (or compression).

⑥ $\vec{F}_{damping} = -\gamma u'$ The damping force is proportional to the first power of the velocity and opposes the direction of motion.

⑦ $\vec{F}_{external} = F(t)$

Using ②-③-④-⑤-⑥-⑦ we have

$$(*) \qquad mu'' = mg - k(u_0+u) - \gamma u' + F(t).$$

Note that at static equilibrium with no external force, we have

$$0 = mu'' = gm - ku_0 - \gamma \cdot 0 \qquad \text{so} \qquad mg - ku_0 = 0 .$$

Substituting this in (*) and rearranging yields

$$\boxed{mu'' + \gamma u' + ku = F(t)}.$$

Note that the equation governing the body's motion is a second order, linear DE with <u>constant coefficients</u>. If $F(t) \not\equiv 0$ then the motion is called <u>forced</u> (or <u>driven</u>). If $F(t) \equiv 0$ then the motion is called <u>free</u>. If $\gamma = 0$ then the motion is called <u>undamped</u>. If $\gamma > 0$ then the motion is called <u>damped</u>.

Ex 1 (#6, p. 202) A mass of 100 grams stretches a spring 5 centimeters. If the mass is set in motion from its equilibrium positive with a downward velocity of 10 centimeters per second, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

Solution: We will use $mu'' + \gamma u' + ku = F(t)$. Note that the problem doesn't mention an external force so we will assume $F(t) = 0$. Also there is no damping so $\gamma = 0$. From the first sentence $m = 100\,g$ and $u_0 = 5$ cm. We use $mg = ku_0$ to determine the stiffness constant of the spring:

$$k = \frac{mg}{u_0} = \frac{(100)(980)}{5} = 19600 \ \frac{\text{dynes-sec}}{\text{cm}}$$

Therefore

$$\boxed{100u'' + 19600u = 0, \quad u(0) = 0, \ u'(0) = 10}$$

is the IVP that describes the body's motion. Letting $u = e^{rt}$ in the DE yields $100r^2 + 19600 = 0$ so $r = \pm\sqrt{-196} = \pm 14i$. Therefore

$$u(t) = c_1\cos(14t) + c_2\sin(14t)$$

is the general solution of the DE. Note that $u'(t) = -14c_1\sin(14t) + 14c_2\cos(14t)$ so $0 = u(0) = c_1$ and $10 = u'(0) = 14c_2$. Therefore

$$\boxed{u(t) = \frac{5}{7}\sin(14t)}$$

gives the position u of the mass at any time $t \geq 0$. Here u is in centimeters and time t is in seconds. To find the time t_1 when the mass first returns to its equilibrium position we solve

$$0 = u(t_1) = \frac{5}{7}\sin(14t_1).$$

Then $14t_1 = \pi$ or $\boxed{t_1 = \frac{\pi}{14} \text{ seconds}}$ (approximately 0.224 seconds)

Ex 2| (#10, p. 203) A mass weighing 16 pounds stretches a spring 3 inches. The mass is attached to a viscous damper with a damping constant of 2 pound-seconds per foot. If the mass is set in motion from its equilibrium position with a downward velocity of 3 inches per second, find its position u at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|u(t)| < 0.01$ inches for all $t > \tau$.

Solution: We use $mu'' + \gamma u' + ku = F(t)$. Since the problem doesn't mention any external force on the mass, we assume $F(t) = 0$. Using

$$mg = \text{weight}$$

we find that $m = \dfrac{16 \text{ lb}}{32 \text{ ft/sec}^2} = \dfrac{1}{2}$ slug. Also, using $mg = ku_0$ we see that

$$k = \frac{16 \text{ lb}}{u_0} = \frac{16 \text{ lb}}{1/4 \text{ ft}} = 64 \text{ lb/ft}$$

The second sentence of the problem description tells us that $\gamma = 2$ lb-s/ft. Therefore

$$\boxed{\frac{1}{2}u'' + 2u' + 64u = 0, \quad u(0) = 0, \; u'(0) = \frac{1}{4} \text{ ft/s}}$$

is the IVP that models the motion of the mass. Then $u = e^{rt}$ in the DE leads to $\frac{1}{2}r^2 + 2r + 64 = 0$ or $r^2 + 4r + 128 = 0$. The quadratic formula yields

$$r = \frac{-4 \pm \sqrt{16 - 4(128)}}{2} = \frac{-4 \pm 4i\sqrt{31}}{2} = -2 \pm 2i\sqrt{31}.$$

The solution to the DE is $e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t)) = u(t)$ where $\lambda = -2$ and $\mu = 2\sqrt{31}$. Consequently

$$u(t) = e^{-2t}\left(c_1 \cos(2\sqrt{31}\,t) + c_2 \sin(2\sqrt{31}\,t)\right)$$

is the general solution of the DE. Note that

$$u'(t) = -2e^{-2t}\left(c_1\cos(2\sqrt{31}\,t) + c_2\sin(2\sqrt{31}\,t)\right) + e^{-2t}\left(-2\sqrt{31}\,c_1\sin(2\sqrt{31}\,t) + 2\sqrt{31}\,c_2\cos(2\sqrt{31}\,t)\right)$$

Therefore $0 = u(0) = c_1$ and $\frac{1}{4} = u'(0) = -2c_1 + 2\sqrt{31}\,c_2$ so $c_2 = \dfrac{1}{8\sqrt{31}}$.

Thus $\boxed{u(t) = \dfrac{1}{8\sqrt{31}} e^{-2t} \sin(2\sqrt{31}\,t)}$ gives the position u of the mass at time t.

(Here u is in feet and t is in seconds.) Etc.