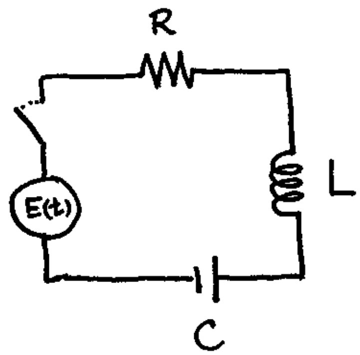
## Sec. 3.8: Forced Oscillations

HW p.215: # 5, 7, 11, 16     Due: Wed., Oct. 6

Schaum's: pp. 114-130.

Consider the RCL series circuit below.




If $Q(t)$ denotes the charge on the capacitor at time $t$ then the equation governing $Q$ is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Discuss the analogy with $mu'' + \gamma u' + ku = F(t)$. Then $L$ is the electrical "mass", $R$ is the electrical "damping constant", and $\frac{1}{C}$ is the electrical "stiffness constant".

For a derivation of this equation using Kirchoff's Laws, see p.201 of Boyce and DiPrima.

**Ex 1** (#16, p.216) A series circuit has a capacitor of $0.25\times10^{-6}$ Farads, a resistor of $5\times10^{3}$ ohms, and an inductor of 1 Henry. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t=0$, determine the charge on the capacitor at $t=0.001$ second, at $t=0.01$ second, and at any time $t$. Also determine the limiting charge as $t\to\infty$.

Solution: We use $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$. In our case $L = 1$ H,

$R = 5000\,\Omega$, and $\frac{1}{C} = \frac{1}{0.25\times 10^{-6}} = 4{,}000{,}000$ F. From the third sentence in the problem description, $E(t) = \text{constant} = 12$ V. Therefore

$$\boxed{\frac{d^2Q}{dt^2} + 5000\frac{dQ}{dt} + 4{,}000{,}000\,Q = 12, \quad Q(0)=0, \quad Q'(0)=0}$$

is the IVP that models the charge on the capacitor. Letting $Q = e^{rt}$ in the corresponding hom. DE leads to $r^2 + 5000r + 4{,}000{,}000 = 0$ or $(r+1000)(r+4000) = 0$ so $r = -1000$ or $r = -4000$. Consequently

$$Q_c(t) = c_1 e^{-1000t} + c_2 e^{-4000t}$$

is the general solution to the associated homogeneous DE. Examining the right member of the nonhomogeneous DE, we see that $Q_p(t) = A$ is a trial solution to the nonhomogeneous DE; here A is a constant to be determined. We want

$$Q_p'' + 5000Q_p' + 4{,}000{,}000Q_p = 12$$

so substituting $Q_p = A$, $Q_p' = 0$, and $Q_p'' = 0$ we have

$$0 + 0 + 4{,}000{,}000\,A = 12$$

so $A = 3\times 10^{-6}$. That is,

$$Q_p(t) = 3\times 10^{-6}.$$

The general solution of the nonhomogeneous DE is $Q = Q_c + Q_p$ so $Q(t) = c_1 e^{-1000t} + c_2 e^{-4000t} + 3\times 10^{-6}$. Then $Q'(t) = -1000c_1 e^{-1000t} - 4000c_2 e^{-4000t}$. Hence $0 \overset{①}{=} Q(0) = c_1 + c_2 + 3\times 10^{-6}$ and $0 = Q'(0) \overset{②}{=} -1000c_1 - 4000c_2$. Multiplying ① by 1000 and adding the result to ② yields $c_2 = 10^{-6}$. Substituting this in ① produces $c_1 = -4\times 10^{-6}$. Consequently, the charge on the capacitor at any time t is $\boxed{Q(t) = -4\times 10^{-6} e^{-1000t} + 10^{-6} e^{-4000t} + 3\times 10^{-6}}$.

Clearly $\boxed{Q(t) \to 3\times 10^{-6} \text{ Coulombs as } t\to\infty.}$ Etc.

Ex 2 (Beats and Resonance; Similar to #10 and #18, pp. 215-216) A body that weighs 8 pounds stretches a spring 6 inches. The undamped system is acted upon by an external force of $8\cos(\omega t)$ pounds where $\omega$ is a positive constant. If the body is released from equilibrium position, determine its displacement from static equilibrium at any positive time t seconds. Sketch the motion when $\omega = 7.8$ and when $\omega = 8$.

Solution: We use $mu'' + \gamma u' + ku = F(t)$. From the second sentence of the problem description, $\gamma = 0$ and $F(t) = 8\cos(\omega t)$. From the equation $mg = \text{weight}$, we have $m = \frac{8\text{ lb.}}{32\text{ ft/sec}^2} = \frac{1}{4}$ slug. Using $mg = ku_0$, we find the stiffness constant of the spring to be $k = \frac{mg}{u_0} = \frac{8\text{ lb.}}{\frac{1}{2}\text{ ft.}} = 16$ lb/ft.

Therefore

$$\boxed{\frac{1}{4}u'' + 16u = 8\cos(\omega t), \quad u(0) = 0,\ u'(0) = 0}$$

models the motion of the body. Letting $u = e^{rt}$ in the associated homogeneous DE, $\frac{1}{4}u'' + 16u = 0$, leads to $\frac{1}{4}r^2 + 16 = 0$ so $r = \pm\sqrt{-64} = \pm 8i$.

Consequently

$$u_c(t) = c_1\cos(8t) + c_2\sin(8t)$$

is the general solution of the associated homogeneous DE.

Case 1: $\omega \neq 8$.

Since $8\cos(\omega t)$ is not a solution of the associated homogeneous equation $\frac{1}{4}u'' + 16u = 0$, the method of undetermined coefficients suggests a trial particular solution of the form $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$ where A and B are constants to be determined. Note that $u_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ and $u_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$. We want

$$\frac{1}{4}u_p'' + 16u_p = 8\cos(\omega t),$$

so substituting the above expressions for $u_p''$ and $u_p$ gives

$$\frac{1}{4}\left[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] + 16\left[A\cos(\omega t) + B\sin(\omega t)\right] = 8\cos(\omega t)$$

or

$$-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) + 64A\cos(\omega t) + 64B\sin(\omega t) = 32\cos(\omega t)$$

or

$$A(64-\omega^2)\cos(\omega t) + B(64-\omega^2)\sin(\omega t) = 32\cos(\omega t) + 0\sin(\omega t).$$

Consequently, equating like coefficients yields $A = \frac{32}{64-\omega^2}$ and $B = 0$. That is,

$$u_p(t) = \frac{32}{64-\omega^2}\cos(\omega t).$$

In this case, the general solution $u = u_c + u_p$ is

$$u(t) = c_1\cos(8t) + c_2\sin(8t) + \frac{32}{64-\omega^2}\cos(\omega t).$$

Note that

$$u'(t) = -8c_1\sin(8t) + 8c_2\cos(8t) - \frac{32\omega}{64-\omega^2}\sin(\omega t)$$

so $0 = u(0) = c_1 + \frac{32}{64-\omega^2}$ and $0 = u'(0) = 8c_2$. Thus

$$\boxed{u(t) = \frac{32}{64-\omega^2}\left[\cos(\omega t) - \cos(8t)\right]} \quad (\omega \neq 8).$$

Using the identity $\cos(A) - \cos(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ this can be written as

$$u(t) = \underbrace{\frac{64}{64-\omega^2}\sin\left(\left(\frac{8-\omega}{2}\right)t\right)}_{\text{slowly varying amplitude when } \omega \text{ is close to } 8.}\sin\left(\left(\frac{8+\omega}{2}\right)t\right).$$

This solution exhibits the phenomenon of "beats" when the driver frequency $\omega$ is close, but not equal, to the natural frequency 8 of the freely oscillating system. For example, when $\omega = 7.8$ we have

$$u(t) \doteq 20.25\sin(0.1t)\sin(7.9t).$$

(Display graph appearing on next page.)

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> f:=20.2532*sin(.1*t)*sin(7.9*t);
>
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$$f := 20.2532\sin(0.1\,t)\sin(7.9\,t)$$

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> plot(f(t),t=-1..63);
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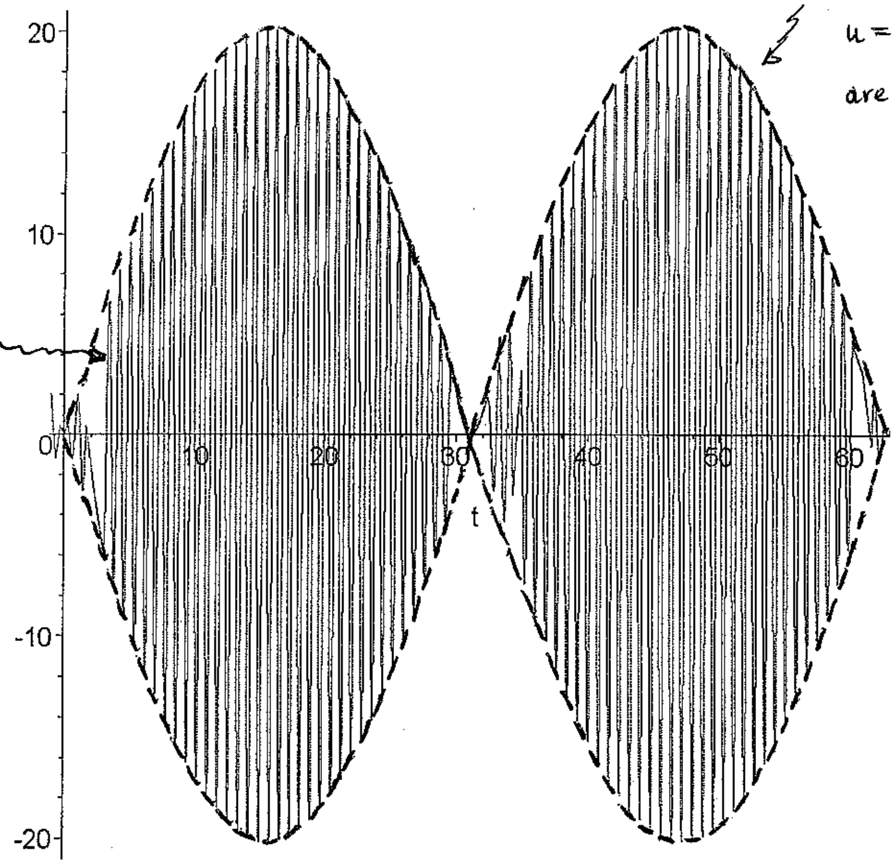
"Envelopes"

$u = \pm 20.25\sin(0.1t)$ are bounded

Solution $u = u(t)$




# Beats

$\sin(0.1t)$ has period $\frac{2\pi}{0.1} = 20\pi$.

$\sin(7.9t)$ has period $\frac{2\pi}{7.9} \doteq \frac{2\pi}{8} = \frac{\pi}{4}$

Therefore the solution oscillates nearly 40 times for each "arch" of the slowly varying envelope cur es.

Case 2: $\omega = 8$.

Since $8\cos(\omega t) = 8\cos(8t)$ is a solution of the associated homogeneous equation $\frac{1}{4}u'' + 16u = 0$, the method of undetermined coefficients suggests a trial particular solution of the form $u_p(t) = t\big(A\cos(8t) + B\sin(8t)\big)$ where A and B are constants to be determined. (See table 3.5.1 on p. 181; we have $P_n(t) = 8$, $e^{\alpha t} = e^{0t} = 1$, and $\cos(\beta t) = \cos(8t)$ so $s = 1$.) Then

$$u_p' = A\cos(8t) + B\sin(8t) + t\big(-8A\sin(8t) + 8B\cos(8t)\big)$$

$$u_p'' = -8A\sin(8t) + 8B\cos(8t) - 8A\sin(8t) + 8B\cos(8t) + t\big(-64A\cos(8t) - 64B\sin(8t)\big)$$
$$= -16A\sin(8t) + 16B\cos(8t) - 64t\big(A\cos(8t) + B\sin(8t)\big).$$

We want $\frac{1}{4}u_p'' + 16u_p = 8\cos(8t)$, so substituting from above yields

$$\frac{1}{4}\Big(-16A\sin(8t) + 16B\cos(8t) - \cancel{64t\big(A\cos(8t)+B\sin(8t)\big)}\Big) + \cancel{16t\big(A\cos(8t)+B\sin(8t)\big)} = 8\cos(8t)$$

or

$$-4A\sin(8t) + 4B\cos(8t) = 0\cdot\sin(8t) + 8\cos(8t).$$

Equating like coefficients produces $A = 0$ and $B = 2$. That is,

$$u_p(t) = 2t\sin(8t).$$

In this case, the general solution $u = u_c + u_p$ is

$$u(t) = c_1\cos(8t) + c_2\sin(8t) + 2t\sin(8t).$$

Note that

$$u'(t) = -8c_1\sin(8t) + 8c_2\cos(8t) + 2\sin(8t) + 16t\cos(8t)$$

so $0 = u(0) = c_1$ and $0 = u'(0) = 8c_2$. Consequently

$$\boxed{u(t) = 2t\sin(8t)}$$

Notice that this solution is not bounded as $t \to \infty$. This solution exhibits the phenomenon of "resonance". Resonance occurs when the driver frequency (8 in this case) equals the natural frequency (8 in this case) of the freely oscillating system.

(Display graph appearing on next page.)

(Mention the Tacoma Narrows Bridge film strip.)

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> g:=2*t*sin(8*t);
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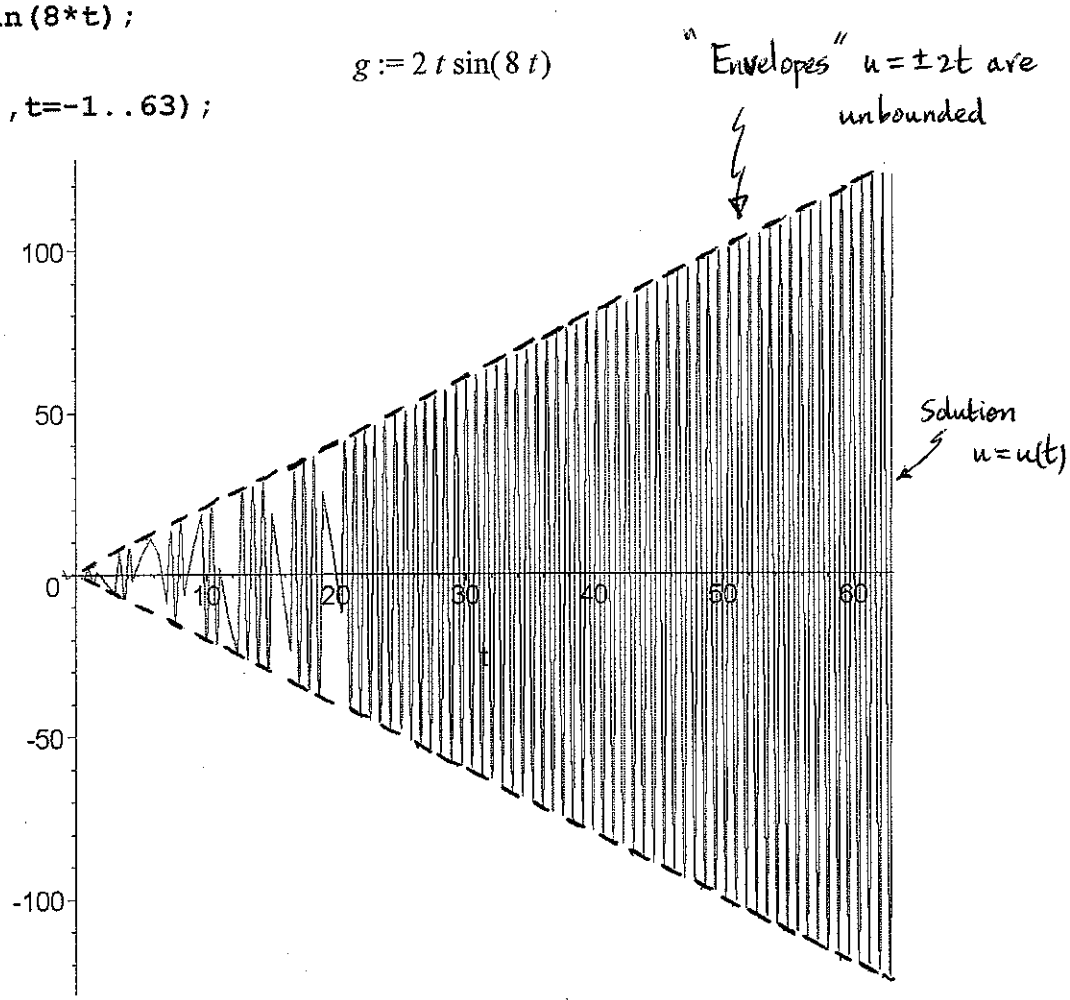
$$g := 2\,t\,\sin(8\,t)$$

```
> plot(g(t),t=-1..63);
```




Resonance