Sec. 4.1 General Theory of n^{th} order Linear Equations
HW p. 224: # 3, 7, 15, 19   Due: Fri., Oct. 8
Schaum's: pp. 89–93

The theory associated with the general n^{th} order linear equation

$$(2) \qquad \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

is very similar to that for second order linear equations which we studied in Sec. 3.2.

Q: When is a linear n^{th} order IVP consisting of $\overbrace{(2)}^{\text{the DE}}$ and the $\overbrace{\text{initial conditions}}^{n}$

$$(3) \qquad y(t_0) = y_0,\ y'(t_0) = y_1,\ \ldots,\ y^{(n-1)}(t_0) = y_{n-1}$$

guaranteed to have exactly one solution?

A: Theorem 4.1.1, p. 220 (Existence and Uniqueness Theorem)

Don't write on board. Ask students to read along on p. 220.

If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = y(t)$ of the DE (2) that also satisfies the initial conditions (3). This solution exists throughout the interval I.

Ex 1 (#5, p. 224) Determine intervals in which solutions to

$$(t-1)y^{(4)} + (t+1)y'' + \tan(t)y = 0$$

are sure to exist.

Solution: In order to make use of the existence/uniqueness theorem 4.1.1 we must place the DE in standard form:

$$y^{(4)} + p_1(t)y^{(3)} + p_2(t)y'' + p_3(t)y' + p_4(t)y = g(t).$$

Our DE is equivalent to

$$y^{(4)} + \frac{t+1}{t-1} y'' + \frac{\tan(t)}{t-1} y = 0.$$

$\therefore$ $p_1(t) = 0$ is continuous on $(-\infty, \infty)$;

$p_2(t) = \dfrac{t+1}{t-1}$ is continuous on $(-\infty, 1)$ and $(1, \infty)$;

$p_3(t) = 0$ is continuous on $(-\infty, \infty)$;

$p_4(t) = \dfrac{\tan(t)}{t-1}$ is continuous on $(-\pi/2, 1)$ and $(1, \pi/2)$ and $(\pi/2, 3\pi/2)$ etc.;

$g(t) = 0$ is continuous on $(-\infty, \infty)$.

The DE is guaranteed to have solutions on any of the following intervals:

$$\ldots, \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, 1\right), \left(1, \pi/2\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

Q2: What is the form of the most general solution of the linear, homogeneous n^{th} order DE

$$(2) \qquad \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t) y = g(t) \ ?$$

Definition: If $f_1, f_2, \ldots, f_n$ are $(n-1)$-times differentiable functions then their Wronskian is the function

$$W(f_1, f_2, \ldots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \ldots & f_n(t) \\ f_1'(t) & f_2'(t) & \ldots & f_n'(t) \\ \vdots & & & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t), & \ldots & f_n^{(n-1)}(t) \end{vmatrix}.$$

Ex 2 (#14, p. 224) Verify that the functions

$$f_1(t) = 1, \quad f_2(t) = t, \quad f_3(t) = e^{-t}, \quad f_4(t) = te^{-t}$$

are solutions of the DE

$$y^{(4)} + 2y''' + y'' = 0,$$

and determine their Wronskian.

Solution: Clearly $f_1' = f_1'' = f_1''' = f_1^{(4)} = 0$ so $f_1^{(4)} + 2f_1''' + f_1'' = 0$.

A similar argument shows f_2 satisfies $f_2^{(4)} + 2f_2''' + f_2'' = 0$. Also

$$f_3^{(4)} + 2f_3''' + f_3'' = e^{-t} + 2(-e^{-t}) + e^{-t} \overset{\checkmark}{=} 0.$$

$$f_4^{(4)} + 2f_4''' + f_4'' = (t-4)e^{-t} + 2(3-t)e^{-t} + (t-2)e^{-t}$$

$$= (\cancel{t} - \cancel{4} + \cancel{6} - \cancel{2t} + \cancel{t} - \cancel{2})e^{-t}$$

$$\overset{\checkmark}{=} 0$$

$$W(f_1, f_2, f_3, f_4)(t) = \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & (1-t)e^{-t} \\ 0 & 0 & e^{-t} & (t-2)e^{-t} \\ 0 & 0 & -e^{-t} & (3-t)e^{-t} \end{vmatrix}$$

(Add row 3 to row 4.)

$$\overset{1R_3 + R_4}{=} \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & (1-t)e^{-t} \\ 0 & 0 & e^{-t} & (t-2)e^{-t} \\ 0 & 0 & 0 & e^{-t} \end{vmatrix}$$

(The determinant of an upper triangular matrix is the product of the elements on the principal diagonal.)

$$= (1)(1)(e^{-t})e^{-t} = \boxed{e^{-2t}}$$

A2: The most general solution of

(2) $$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

on an interval I is of the form

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + Y(t),$$

where ⓐ $y_1, \ldots, y_n$ is a set of n solutions to the associated homogeneous equation of order n,

(4) $$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0,$$

such that $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ on I;

I.e. $y_1, y_2, \ldots, y_n$ is a F.S.S. for (4).

ⓑ $c_1, c_2, \ldots, c_n$ are arbitrary constants;

ⓒ $y = Y(t)$ is a particular solution of (2) on I.

(See Theorem 4.1.2, p.221 and the discussion on p.224)

Ex 3 (Similar to #19, p.225) Consider the linear operator

$$L[y] = y^{(4)} - 10y'' + 9y$$

(a) Find $L[e^{rt}]$.

(b) Determine four solutions of the DE: $y^{(4)} - 10y'' + 9y = 0$. Do you think the four solutions form a F.S.S.? Why?

Solution: (a) $$L[e^{rt}] = (e^{rt})^{(4)} - 10(e^{rt})'' + 9e^{rt}$$
$$= r^4e^{rt} - 10r^2e^{rt} + 9e^{rt}$$
$$= (r^4 - 10r^2 + 9)e^{rt}$$

(b) If we let $y = e^{rt}$ in $y^{(4)} - 10y'' + 9y = 0$ then by (a),

$(r^4 - 10r^2 + 9)e^{rt} = 0$ so $r^4 - 10r^2 + 9 = 0$ or $(r^2-1)(r^2-9) = 0$ or $(r-1)(r+1)(r+3)(r-3) = 0$ so $r = 1, -1, 3,$ or -3. Thus, four solutions are

$$\boxed{y_1(t) = e^{t},\quad y_2(t) = e^{-t},\quad y_3(t) = e^{3t},\quad y_4(t) = e^{-3t}.}$$

Note: $W(y_1, y_2, y_3, y_4)(t) = \begin{vmatrix} e^{t} & e^{-t} & e^{3t} & e^{-3t} \\ e^{t} & -e^{-t} & 3e^{3t} & -3e^{-3t} \\ e^{t} & e^{-t} & 9e^{3t} & 9e^{-3t} \\ e^{t} & -e^{-t} & 27e^{3t} & -27e^{-3t} \end{vmatrix}$ (Factor e^{t} out of the first column, e^{-t} out of the second, e^{3t} out of the third, and e^{-3t} out of the fourth.)

$$= \underbrace{e^{t} \cdot e^{-t} \cdot e^{3t} \cdot e^{-3t}}_{1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -3 \\ 1 & 1 & 9 & 9 \\ 1 & -1 & 27 & -27 \end{vmatrix}$$

(Add -1 times row 1 to row 3. Add -1 times row 2 to row 4.)

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -3 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 24 & -24 \end{vmatrix}$$

(Add -1 times row 1 to row 2, Add -3 times row 3 to row 4.)

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & -48 \end{vmatrix}$$

(The determinant of an upper triangular matrix is the product of the elements on the principal diagonal.)

$$= (1)(-2)(8)(-48)$$

$$= 768 \ (\neq 0).$$

$$\boxed{\text{Therefore } y_1, y_2, y_3, y_4 \text{ form a F.S.S. to } y^{(4)} - 10y'' + 9y = 0.}$$

The general solution is $y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{3t} + c_4 e^{-3t}$ where c_1, c_2, c_3, c_4 are arbitrary constants.