Sec. 4.2 Homogeneous Equations with Constant Coefficients

HW p. 231: # 11, 22, 29, 39 Due: Mon., Oct. 11

Schaum's: pp. 89–93

To solve

n^{th} order, linear, homogeneous DE with constant coefficients.

$$(*) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real constants, we assume $y = e^{rt}$. This leads to

$$a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} = 0,$$

and dividing through by e^{rt} yields

$$(\dagger) \qquad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Characteristic Equation of $(*)$.

Distinct real roots

Ex 1 (#13, p. 232) Find the general solution of $2y''' - 4y'' - 2y' + 4y = 0$.

Soln: $y = e^{rt}$ leads to $2r^3 - 4r^2 - 2r + 4 = 0$

$$1 r^3 - 2r^2 - r + 2 = 0$$

The possible rational zeros are of the form $\frac{p}{q}$ where p divides 2 and q divides 1. Therefore $\pm 1, \pm 2$ are the possible rational zeros. We see that $r = -1$ is a zero by inspection, so $r+1$ is a divisor of the polynomial.

$$\begin{array}{r l}
 & r^2 - 3r + 2 \\
r+1 \,\big) & r^3 - 2r^2 - r + 2 \\
 & -(r^3 + r^2) \\
 & \quad -3r^2 - r \\
 & \quad -(-3r^2 - 3r) \\
 & \qquad 2r + 2 \\
 & \qquad -(2r + 2) \\
 & \qquad\quad 0 = R
\end{array}$$

Thus the characteristic equation is $(r+1)(r^2 - 3r + 2) = 0$

Factoring the quadratic yields $(r+1)(r-2)(r-1)=0$ so $r=-1, 2,$ or 1.
Therefore $y_1 = e^{-t}$, $y_2 = e^{2t}$, $y_3 = e^{t}$ are solutions to the DE. To check if they form a F.S.S. we compute their Wronskian:

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} e^{-t} & e^{2t} & e^{t} \\ -e^{-t} & 2e^{2t} & e^{t} \\ e^{-t} & 4e^{2t} & e^{t} \end{vmatrix}$$

(Factor e^{-t} from column 1, e^{2t} from column 2, and e^{t} from column 3.)

$$= e^{-t} \cdot e^{2t} \cdot e^{t} \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

(Add row 1 to row 2, Add -1 times row 1 to row 3.)

$$= e^{2t} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 0 \end{vmatrix}$$

(Expand by cofactors along row 3.)

$$= e^{2t}(-1)(3) \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= -6e^{2t} \neq 0$$

Therefore $y_1 = e^{-t}$, $y_2 = e^{2t}$, $y_3 = e^{t}$ form a F.S.S. to the DE. The general solution is

$$\boxed{y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{t}}$$

where $c_1, c_2,$ and c_3 are arbitrary constants.

Ex 2 (Similar to #18, p.232) Find the general solution of $y^{(4)}-4y'''+10y''-12y'+5y=0$

Soln: $y=e^{rt}$ leads to $r^4-4r^3+10r^2-12r+5=0$. The possible rational zeros are $\frac{p}{q}$ where p divides 5 and q divides 1. Therefore $\pm 1, \pm 5$ are the candidates for rational zeros. By inspection we see that $r=1$ is a zero. We use synthetic division to examine the multiplicity of $r=1$ as a zero.

$$
\begin{array}{r|rrrrr}
1 & 1 & -4 & 10 & -12 & 5 \\
 & & 1 & -3 & 7 & -5 \\
\hline
1 & 1 & -3 & 7 & -5 & 0 = R \\
 & & 1 & -2 & 5 & \\
\hline
 & 1 & -2 & 5 & 0 = R &
\end{array}
$$

$$(r-1)(r^3-3r^2+7r-5)=0$$

$$(r-1)(r-1)(r^2-2r+5)=0$$

We use the quadratic formula to finish: $r = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

The roots of the characteristic equation are:

$$r = 1 \text{ (multiplicity two)}, \quad r = 1 \pm 2i.$$

Therefore $y_1 = e^t$, $y_2 = te^t$, $y_3 = e^t\cos(2t)$, $y_4 = e^t\sin(2t)$ are solutions of the DE. One checks that

$$W(y_1, y_2, y_3, y_4)(t) = 16e^{4t} \neq 0 \qquad \text{(cf. \#20(d), p. 225)}$$

Therefore the set is a F.S.S. The general solution is

$$\boxed{y = c_1 e^t + c_2 t e^t + e^t\left(c_3\cos(2t) + c_4\sin(2t)\right)}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.