Sec. 4.3: The Method of Undetermined Coefficients

HW p. 237: # 2, 11, 15 Due: Wed., Oct. 13

Schaum's: pp. 94 – 102 (esp. 11.7)

Sec. 4.4: Variation of Parameters Due: Fri, Oct. 15

HW p. 242: # 1 (on the interval $-\pi/2 < t < \pi/2$), 9, 13

Schaum's: pp. 103 – 109.

In these two sections we will address the problem of solving the n^{th} order linear nonhomogeneous equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t).$$

We will concentrate mainly on the case when the coefficients are constant.

Ex 1 (#2, p. 237) Find the general solution of

$$y^{(4)} - y = 3t + \cos(t).$$

Solution:

Step ①: $y = e^{rt}$ in $y^{(4)} - y = 0$ leads to $r^4 - 1 = 0$. Factoring yields $(r^2-1)(r^2+1) = 0$ or $(r-1)(r+1)(r^2+1) = 0$. Therefore the characteristic equation has roots: $r = 1, -1, i, -i$. The general solution of the associated homogeneous equation is $y_c(t) = c_1 e^t + c_2 e^{-t} + c_3\cos(t) + c_4\sin(t)$.

Step ②: We will use the method of undetermined coefficients to find a particular solution of the nonhomogeneous equation. Since $g(t) = 3t + \cos(t)$, we would normally use a trial form $y_p(t) = At + B + C\cos(t) + D\sin(t)$. However $\cos(t)$ and $\sin(t)$ are already solutions to the associated homogeneous equation, so we must multiply $C\cos(t)+D\sin(t)$ by t. (That is, $s=1$ in the table 3.5.1 on p. 181.) Therefore we use a trial form

$$y_p(t) = At+B+t\big(C\cos(t)+D\sin(t)\big)$$

where A, B, C, and D are constants to be determined. Then

$$y_p' = A + C\cos(t)+D\sin(t) + t\big(-C\sin(t)+D\cos(t)\big)$$

$$y_p'' = 2\big(-C\sin(t)+D\cos(t)\big) + t\big(-C\cos(t)-D\sin(t)\big)$$

$$y_p''' = 3\big(-C\cos(t)-D\sin(t)\big) + t\big(C\sin(t)-D\cos(t)\big)$$

$$y_p^{(4)} = 4\big(C\sin(t)-D\cos(t)\big) + \underbrace{t\big(C\cos(t)+D\sin(t)\big)}_{y_p-(At+B)}$$

We want

$$y^{(4)} - y_p = 3t+\cos(t)$$

so substituting expressions for $y_p^{(4)}$ and y_p from above yields

$$-(At+B)+4\big(C\sin(t)-D\cos(t)\big) = 3t+\cos(t)$$

or

$$-At-B+4C\sin(t)-4D\cos(t) = 3t+0\cdot 1+0\cdot\sin(t)+1\cdot\cos(t).$$

Equating like coefficients gives $-A=3$, $-B=0$, $4C=0$, and $-4D=1$.
Therefore $y_p(t) = -3t-\frac{1}{4}t\sin(t)$.

Step③: Write the general solution $y=y_c+y_p$:

$$\boxed{y(t) = c_1e^{t}+c_2e^{-t}+c_3\cos(t)+c_4\sin(t)-3t-\frac{1}{4}t\sin(t)}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Variation of Parameters: A particular solution of

$$(*) \qquad y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is $\boxed{y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \dots + u_n(t)y_n(t)}$

where $y_1, y_2, \dots, y_n$ is a fundamental set of solutions of the associated homogeneous equation of $(*)$ and

$$\boxed{u_i(t) = \int \frac{g(t)W_i(t)}{W(t)}\,dt} \qquad (i = 1, 2, \dots, n).$$

Here $W(t)$ denotes the Wronskian of $y_1, y_2, \dots, y_n$ at t and $W_i(t)$ is the determinant obtained from $W(t)$ by replacing the column containing $y_i(t)$ and its derivatives by the column $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

<u>Notes</u>: ① If you want to use variation of parameters for y_p then you <u>must</u> remember to place the DE $(*)$ in <u>standard form</u>. That is, the leading coefficient must be 1. (See #13 on p. 242.)

② The variation of parameters formula is derived in the text on pp. 239-240.

<u>Ex 2</u> (#4, p. 242) Find the general solution of

$$y''' + y' = \sec(t)$$

on the interval $-\frac{\pi}{2} < t < \pi/2$.

Solution:

Step ①: $y=e^{rt}$ in $y'''+y'=0$ leads to $r^3+r=0$ or $r(r^2+1)=0$.
Therefore the roots of the characteristic equation are $r=0, i, -i$. Then
$y_1(t)=1$, $y_2(t)=\cos(t)$, $y_3(t)=\sin(t)$ are solutions of $y'''+y'=0$.

Expand by cofactors along column 1.

$$W(t)=\begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t)\end{vmatrix} = 1\cdot\begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t)\end{vmatrix} - 0\begin{vmatrix} \cos(t) & \sin(t) \\ -\cos(t) & -\sin(t)\end{vmatrix} + 0\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{vmatrix}$$

$$=1$$

Therefore y_1, y_2, y_3 is a F.S.S. for $y'''+y'=0$.

Step ②: We use variation of parameters to find y_p because $g(t)=\sec(t)$ is not an exponential, polynomial, or sine or cosine function (or sums or products of such).

Then
$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t)$$
$$= u_1(t) + \cos(t)u_2(t) + \sin(t)u_3(t).$$

Replace $\begin{bmatrix}1\\0\\0\end{bmatrix}$ by this column.

Expand by cofactors along column 1

$$W_1(t) = \begin{vmatrix} 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 1 & -\cos(t) & -\sin(t)\end{vmatrix} = 1\cdot\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{vmatrix} = 1$$

$$\therefore\ u_1(t) = \int \frac{gW_1}{W}\,dt = \int \frac{\sec(t)\cdot 1}{1}\,dt = \ln|\sec(t)+\tan(t)| + \cancel{C}^{\,0}$$

Replace $\begin{bmatrix}\cos(t)\\-\sin(t)\\-\cos(t)\end{bmatrix}$ by this column.

Expand by cofactors along column 1

$$W_2(t) = \begin{vmatrix} 1 & 0 & \sin(t) \\ 0 & 0 & \cos(t) \\ 0 & 1 & -\sin(t)\end{vmatrix} = 1\cdot\begin{vmatrix} 0 & \cos(t) \\ 1 & -\sin(t)\end{vmatrix} = -\cos(t)$$

$$\therefore\ u_2(t) = \int \frac{gW_2}{W}\,dt = \int \frac{\sec(t)(-\cos(t))}{1}\,dt = \int -1\,dt = -t + \cancel{C}^{\,0}$$

Replace $\begin{bmatrix}\sin(t)\\\cos(t)\\-\sin(t)\end{bmatrix}$ by this column.

Expand by cofactors along column 1

$$W_3(t) = \begin{vmatrix} 1 & \cos(t) & 0 \\ 0 & -\sin(t) & 0 \\ 0 & -\cos(t) & 1\end{vmatrix} = 1\cdot\begin{vmatrix} -\sin(t) & 0 \\ -\cos(t) & 1\end{vmatrix} = -\sin(t)$$

$$\therefore \quad u_3(t) = \int \frac{g W_3}{W}\,dt = \int \frac{\sec(t)(-\sin(t))}{1}\,dt = \int \frac{-\sin(t)\,dt}{\cos(t)}$$

Let $v = \cos(t)$
Then $dv = -\sin(t)\,dt$

$$= \int \frac{dv}{v} = \ln|v| + \cancel{C}^{\,0} = \ln|\cos(t)|$$

Consequently, $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$= \ln|\sec(t) + \tan(t)| - t\cos(t) + \sin(t)\ln|\cos(t)|.$$

Step③: Write the general solution $y = y_c + y_p$:

$$\boxed{y(t) = c_1 + c_2\cos(t) + c_3\sin(t) + \ln|\sec(t) + \tan(t)| - t\cos(t) + \sin(t)\ln|\cos(t)|}$$

where c_1, c_2, c_3 are arbitrary constants.