## Sec. 5.4: Euler Equations

HW p. 276: #1, 5, 14, 35   Due: Fri., Oct. 15

Schaum's p. 287 (#28.22 & 28.23) & p. 289 (#28.34 – 28.38)

Chapter 5 is devoted to solving

(□) $$p_2(t)y'' + p_1(t)y' + p_0(t)y = 0$$

where $p_2$, $p_1$, and $p_0$ are polynomial functions of $t$. Here are some important examples:

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0 \qquad \text{(Bessel's equation of order } \nu\text{)}$$

$$(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0 \qquad \text{(Legendre's " " " } \alpha\text{)}$$

$$ty'' + (1-t)y' + \lambda y = 0 \qquad \text{(Laguerre's " " " } \lambda\text{)}$$

$$y'' - 2ty' + \lambda y = 0 \qquad \text{(Hermite's " " " } \lambda\text{)}$$

In many cases the solutions to (□) are given by power series expressions:

$$y(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n \quad \text{or} \quad y(t) = t^{r_0}\sum_{n=0}^{\infty} c_n(t-t_0)^n.$$

(See Theorems 5.3.1 (pp. 262-3) and 5.6.1 (pp. 289-90).)

In Sec. 5.4 we look at the special case

(*) $$\boxed{at^2y'' + bty' + cy = 0} \qquad (a, b, c \text{ real constants}, a \neq 0)$$

which was first studied by Leonhard Euler. Following Euler's lead, we will look for power function solutions to (*): $y(t) = t^m$, where $m$ is a constant. Then $y' = mt^{m-1}$ and $y'' = m(m-1)t^{m-2}$ so substituting in (*) gives

$$at^2 m(m-1)t^{m-2} + btmt^{m-1} + ct^m = 0.$$

But $t^2 \cdot t^{m-2} = t^m$ and $t \cdot t^{m-1} = t^m$ so

$$am(m-1)t^m + bmt^m + ct^m = 0 \quad \text{or}$$

$$(\dagger) \qquad \boxed{am(m-1) + bm + c = 0}.$$

The algebraic equation (†) is called the <u>characteristic equation</u> of the DE (*).

<u>Ex 1</u> (#12, p. 276) [Distinct Real Roots]. Solve $t^2y'' - 4ty' + 4y = 0$.

<u>Soln:</u> $y = t^m$ in the DE leads to $m(m-1) - 4m + 4 = 0 \Rightarrow$

$m^2 - 5m + 4 = 0 \Rightarrow (m-4)(m-1) = 0$ so $m = 1$ or $m = 4$.

$y_1(t) = t$ and $y_2(t) = t^4$ are solutions to the DE.

$$W(y_1, y_2)(t) = \begin{vmatrix} t & t^4 \\ 1 & 4t^3 \end{vmatrix} = 3t^4 \neq 0 \quad \text{if} \quad t \neq 0.$$

Therefore $\boxed{y = c_1 t + c_2 t^4}$ is the general solution of the DE on $(-\infty, 0)$ and on $(0, \infty)$ where $c_1, c_2$ are arbitrary constants.

<u>Ex 2</u> (#3, p. 276) [Repeated Real Roots]. Solve $t^2y'' - 3ty' + 4y = 0$ on $t > 0$.

<u>Soln:</u> $y = t^m$ in the DE leads to $m(m-1) - 3m + 4 = 0 \Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$ so $m = 2$ (multiplicity two). $y_1 = t^2$ solves the DE on $-\infty < t < \infty$.

To find a second linearly independent solution we use the method of reduction of order: $y_2(t) = u(t)y_1(t) = t^2u(t)$. Then $y_2' = t^2u' + 2tu$ and

$y_2'' = t^2u'' + 2tu' + 2tu' + 2u = t^2u'' + 4tu' + 2u$. Substituting in

$$t^2y_2'' - 3ty_2' + 4y_2 \overset{\text{want}}{=} 0$$

we get

$$t^2(t^2u'' + 4tu' + 2u) - 3t(t^2u' + 2tu) + 4t^2u = 0$$

or

$$t^4u'' + (4t^3 - 3t^3)u' + (2t^2 - 6t^2 + 4t^2)u = 0$$

$$t^4u'' + t^3u' = 0.$$

Let $v = u'$. Then $v' = u''$ so the above equation becomes

$$\underbrace{tv' + v}_{\text{Exact!}} = 0$$

$$\frac{d}{dt}[tv] = 0$$

Integrating yields $tv = c_1$ so $u' = v = \frac{c_1}{t}$. Integrating again produces $u = c_1\ln(t) + c_2$. Thus $y_2(t) = t^2(c_1\ln(t) + c_2) = c_1t^2\ln(t) + c_2t^2$

To get a second linearly independent solution we set $c_1 = 1$ and $c_2 = 0$, so $y_2(t) = t^2\ln(t)$.

If pressed for time, say that this was problem #6 on Exam I and give the solution: $y_2(t) = t^2\ln(t)$.

Check: $W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^2\ln(t) \\ 2t & t + 2t\ln(t) \end{vmatrix} = t^3 \neq 0$ for $t > 0$.

Therefore $\boxed{y(t) = c_1t^2 + c_2t^2\ln(t)}$ ($c_1, c_2$ arbitrary constants) is the general solution of the DE on the interval $0 < t < \infty$.

**Ex 3** (#4, p.276) [Complex Conjugate Roots] Solve $t^2y'' + 3ty' + 5y = 0$ on $t>0$.

Soln: $y = t^m$ leads to $m(m-1) + 3m + 5 = 0 \implies m^2 + 2m + 5 = 0$

$$\implies m = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$ Therefore we have

solutions $y_1(t) = t^{-1+2i}$ and $y_2(t) = t^{-1-2i}$. We need to find a F.S.S.

that is real-valued. To do this we make use of Euler's identity: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

where $\theta$ is real.

$$y_1(t) = t^{-1+2i} = t^{-1} \cdot t^{2i} = t^{-1}\left(e^{\ln t}\right)^{2i} = t^{-1}e^{i2\ln(t)} = t^{-1}\left(\cos(2\ln t) + i\sin(2\ln t)\right)$$

Similarly

$$y_2(t) = t^{-1}\left(\cos(2\ln t) - i\sin(2\ln t)\right)$$

By the superposition principle, the following are also solutions to the homogeneous linear DE:

$$\tilde{y}_1(t) = \mathrm{Re}(y_1(t)) = \frac{y_1(t) + y_2(t)}{2} = t^{-1}\cos(2\ln t)$$

$$\tilde{y}_2(t) = \mathrm{Im}(y_1(t)) = \frac{y_1(t) - y_2(t)}{2i} = t^{-1}\sin(2\ln t)$$

One checks that $W(\tilde{y}_1, \tilde{y}_2)(t) = 2t^{-3} \neq 0$ for $t > 0$. Therefore

$y(t) = c_1 t^{-1}\cos(2\ln t) + c_2 t^{-1}\sin(2\ln t)$ is the general solution of the DE on $0<t<\infty$.

Summary: To solve the Euler equation

$$(*) \qquad at^2y'' + bty' + cy = 0,$$

$y = t^m$ leads to

$$(\dagger) \qquad am(m-1) + bm + c = 0.$$

| Roots of $(\dagger)$ | F.S.S. of $(*)$ on $t>0$ | Gen. Soln. of $(*)$ on $t>0$ |
|---|---|---|
| $m_1, m_2$ distinct, real | $t^{m_1}, \ t^{m_2}$ | $y = c_1 t^{m_1} + c_2 t^{m_2}$ |
| $m = m_1 = m_2$ real, repeated | $t^{m_1}, \ t^{m_1}\ln(t)$ | $y = t^{m_1}(c_1 + c_2 \ln(t))$ |
| $m = \lambda \pm i\mu$ complex conjugates $(\mu \neq 0)$ | $t^{\lambda}\cos(\mu \ln t), \ t^{\lambda}\sin(\mu\ln t)$ | $y = t^{\lambda}(c_1\cos(\mu\ln t) + c_2\sin(\mu\ln t))$ |

Supplementary Problem (time permitting): #3(b) on Exam II, Spring 2010.

Solve $t^2y'' - 2ty' + 2y = t^2$ on $0 < t < \infty$.

<u>Soln</u>: $y(t) = \underbrace{c_1 t + c_2 t^2}_{y_c} + \underbrace{t^2 \ln(t)}_{y_p \text{ (via variation of parameters)}}$ where $c_1, c_2$ are arbitrary constants.