Sec. 6.3 Step Functions

HW p. 328: #11, 13, 20, 24   Due: Wed., Oct. 27

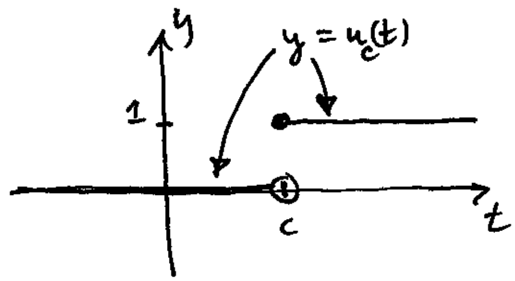
Schaum's: pp. 233, 236f (#23.8 – 23.14), 240–241 (#23.36 – 23.55)

The next tool helps us handle "abruptly changing" drivers in IVP's.

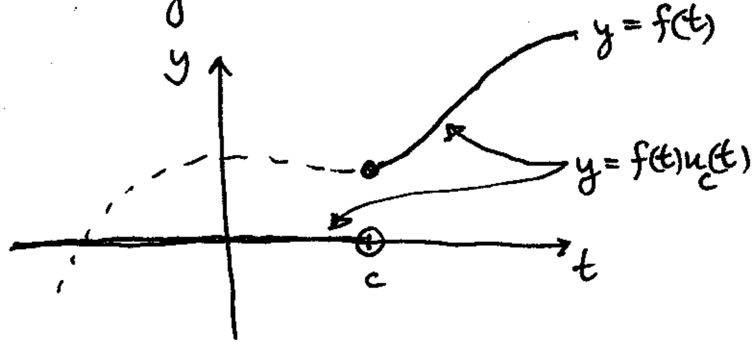
Definition: The unit step function at c is given by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c, \\ 1 & \text{if } t \geq c. \end{cases}$$

The graph of the unit step function looks like this:

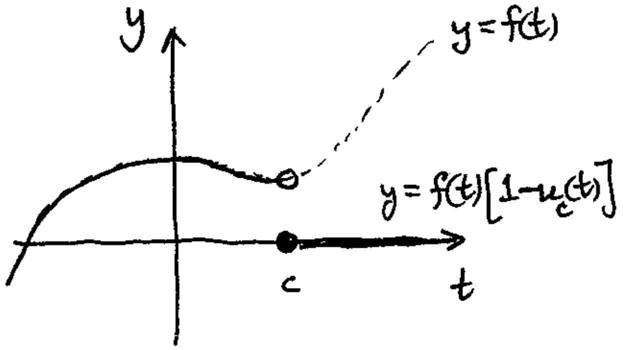


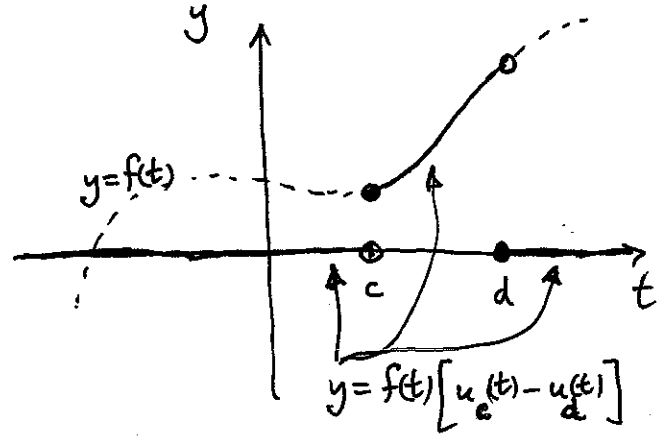
Multiplication of a function by the unit step function at c acts like a "switch" turning on the function at c.



Similarly, multiplication by $1 - u_c(t)$ acts like an "off switch" at c;

" " $u_c(t) - u_d(t)$ " " " "on-off switch".



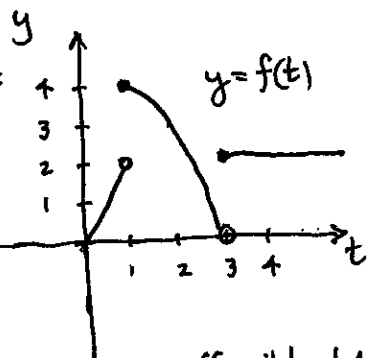


Ex 1 (Similar to #11, p. 329) Write the abruptly changing function

$$f(t) = \begin{cases} 2t & \text{if } 0 \le t < 1, \\ 4-(t-1)^2 & \text{if } 1 \le t < 3, \\ 2 & \text{if } t \ge 3, \end{cases}$$

in terms of the unit step function $u_c(t)$ and sketch the graph of f.

<u>Solution:</u>



$$f(t) = 2t\Big[\overbrace{1-u_1(t)}^{\text{off switch at 1}}\Big] + \left[4-(t-1)^2\right]\Big[\overbrace{u_1(t)-u_3(t)}^{\text{on-off switch; on at 1, off at 3}}\Big] + 2\overbrace{u_3(t)}^{\text{on switch at 3}}$$

$$f(t) = 2t + \left[4-2t-(t-1)^2\right]u_1(t) + \left[-4+(t-1)^2+2\right]u_3(t)$$

$$\boxed{f(t) = 2t + (3-t^2)u_1(t) + (t^2-2t-1)u_3(t)}$$

The tool for taking the Laplace transform of functions involving the unit step function is:

Read from text book. Don't write on board.

<u>Theorem 6.3.1</u> (p.326) If the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and if c is a positive constant then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s) \qquad (*)$$

for $s > a$. Conversely if $f(t) = \mathcal{L}^{-1}\{F(s)\}$ then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \qquad (**)$$

(See p.326 of the text for a proof.)

Here are some special cases of $(*)$:

$[f(t)=1]$ $$\mathcal{L}\{u_c(t)\}(s) = e^{-cs}\mathcal{L}\{1\}(s) = \frac{e^{-cs}}{s}$$

$\left[\begin{matrix} f(t)=t \\ c=2 \end{matrix}\right]$ $$\mathcal{L}\{(t-2)u_2(t)\}(s) = e^{-2s}\mathcal{L}\{t\}(s) = \frac{e^{-2s}}{s^2}.$$

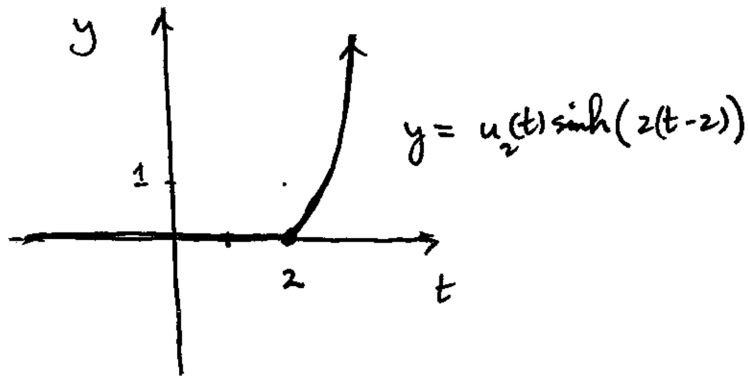
<u>Ex 2</u> (#22, p.329) Compute $\mathcal{L}^{-1}\left\{\dfrac{2e^{-2s}}{s^2-4}\right\}$ and sketch its graph.

<u>Solution</u>: A routine partial fraction decomposition shows that

$$\frac{2}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} = \frac{1/2}{s-2} - \frac{1/2}{s+2}.$$

Therefore
$$\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2-4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\}$$
$$= \frac{1}{2}u_2(t)e^{2(t-2)} - \frac{1}{2}u_2(t)e^{-2(t-2)}$$
$$= \boxed{u_2(t)\sinh(2(t-2))}$$

Apply $(**)$ with $F_1(s) = \dfrac{1}{s-2}$ and $F_2(s) = \dfrac{1}{s+2}$



In order to compute transforms of functions involving unit step functions it is often preferable to use the identity (*) in Theorem 6.3.1 in a different but equivalent form:

$$\mathcal{L}\{u_c(t)g(t)\}(s) = e^{-cs}\mathcal{L}\{g(t+c)\}(s). \qquad (*')$$

<u>Ex 3</u> (Similar to #17, p. 329) Compute $\mathcal{L}\{(3t+1)u_1(t)\}(s)$.

<u>Solution:</u>

$$\mathcal{L}\{\overbrace{(3t+1)}^{g(t)}\overbrace{u_1(t)}^{c=1}\}(s) = e^{-s}\mathcal{L}\{\overbrace{3(t+1)+1}^{g(t+1)}\}(s) \qquad (\text{Apply } (*'))$$

$$= e^{-s}\left(\mathcal{L}\{3t\}(s) + \mathcal{L}\{4\}(s)\right)$$

$$= \boxed{\frac{3e^{-s}}{s^2} + \frac{4e^{-s}}{s}}$$