Sec. 6.4 DEs with Discontinuous Forcing Functions

HW p. 336: # 1, 5, 16 Due: Fri., Oct. 29

Schaum's: pp. 245f [#24.9, 24.33]

In this section we solve IVP's like

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \; y'(0) = y_1$$

where the forcing function g changes abruptly at one or more points.

Ex 1 (Similar to #7, p. 336) Find the solution of the IVP

$$y'' + y = g(t), \quad y(0) = 0, \; y'(0) = -1$$

where $g(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi, \\ 0 & \text{if } \pi \le t. \end{cases}$ Draw the graphs of the solution and the forcing function and explain how they are related.

Solution: We use the method of Laplace transforms. First note that g can be expressed in terms of unit step functions as

$$g(t) = 1 - u_\pi(t).$$

Step 1:

$$\mathcal{L}\{y'' + y\}(s) = \mathcal{L}\{1 - u_\pi(t)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\cancel{y(0)}^{\,0} - \overset{-1}{y'(0)} + \mathcal{L}\{y\}(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

Step 2:

$$(s^2+1)\mathcal{L}\{y\}(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} - 1$$

$$\mathcal{L}\{y\}(s) = \frac{-1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi s}}{s(s^2+1)}$$

Step 3: $y(t) = \mathcal{L}^{-1}\left\{ -\frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi s}}{s(s^2+1)} \right\}$

$$= -\sin(t) + \mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+1)} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-\pi s}}{s(s^2+1)} \right\}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \overset{\text{Routine PFD}}{=} \frac{1}{s} - \frac{s}{s^2+1}.$$

$$\therefore \quad y(t) = -\sin(t) + \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{s}{s^2+1} \right\} - \mathcal{L}^{-1}\left\{ e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1} \right) \right\}$$

$$= -\sin(t) + 1 - \cos(t) - u_\pi(t) f(t-\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{s}{s^2+1} \right\} = 1 - \cos(t)$.

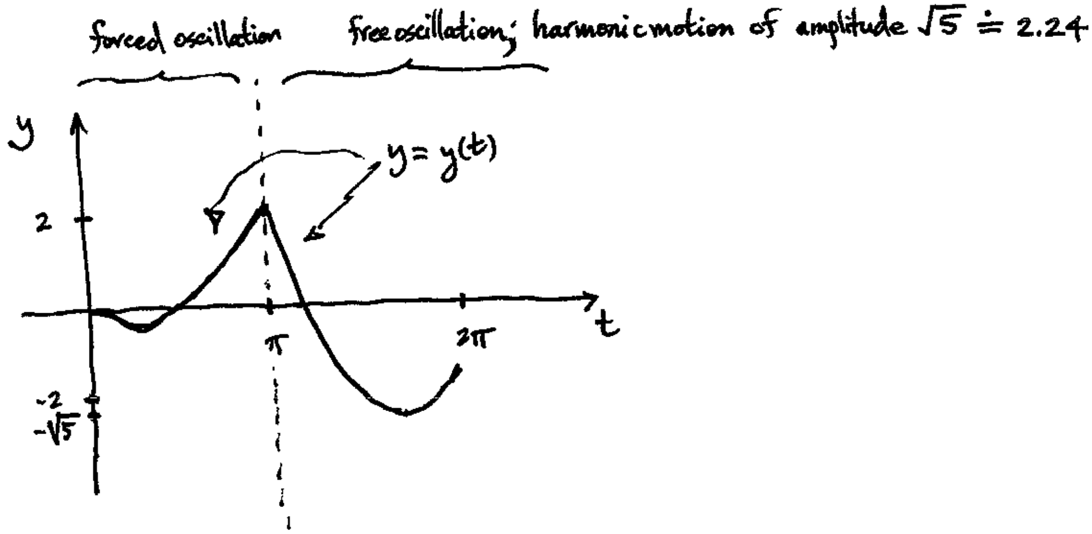
$$\therefore \quad y(t) = 1 - \sin(t) - \cos(t) - u_\pi(t)\left[1 - \cos(t-\pi) \right]$$

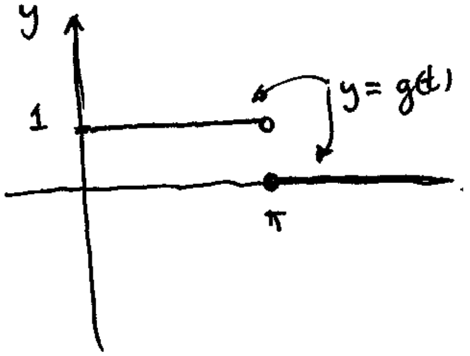
But $\cos(t-\pi) = \cos(t)\cos(\pi) + \sin(t)\sin(\pi) = -\cos(t)$ so

$$\boxed{y(t) = 1 - \sin(t) - \cos(t) - \left[1 + \cos(t)\right] u_\pi(t)}$$

$$\therefore \quad y(t) = \begin{cases} 1 - \sin(t) - \cos(t) & \text{if } 0 \le t < \pi, \\ 1 - \sin(t) - \cos(t) - \left[1 + \cos(t)\right] & \text{if } \pi \le t, \end{cases}$$

$$= \begin{cases} 1 - \sin(t) - \cos(t) & \text{if } 0 \le t < \pi, \\ -\sin(t) - 2\cos(t) & \text{if } \pi \le t. \end{cases}$$


forced oscillation
free oscillation; harmonic motion of amplitude $\sqrt{5} \doteq 2.24$
y
2
-2
$-\sqrt{5}$
π
2π
t
$y = y(t)$


y
1
$y = g(t)$
π