Sec. 6.6 The Convolution Integral

HW p. 350: #4, 9, 14, 22 Due: Wed., Nov. 3

Schaum's: pp. 233f (#23.1 – 23.7, #23.15 – 23.19) (Also #23.20 – 23.35, #23.56 – 23.60)

If f and g are absolutely integrable functions on the real line $(-\infty, \infty)$, then their convolution product $f*g$ is the function on $(-\infty,\infty)$ defined by

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau .$$

This is the "general" definition of convolution product that appears naturally in subjects like signal analysis and partial differential equations. However, in Math 204 we are interested in solving initial value problems forward in time from $t=0$. In such problems it is customary to assume that $f(t) = 0 = g(t)$ for $t<0$. With this assumption, the definition of the convolution product simplifies as follows:

$$(f*g)(t) = \int_{-\infty}^{0} f(t-\tau)\underbrace{g(\tau)}_{\substack{\text{zero since}\\ \tau<0}}\,d\tau + \int_{0}^{t} f(t-\tau)g(\tau)\,d\tau + \int_{t}^{\infty} \underbrace{f(t-\tau)}_{\substack{\text{zero since}\\ t-\tau<0\\ \text{when } \tau>t.}}g(\tau)\,d\tau$$

$$\boxed{(f*g)(t) = \int_{0}^{t} f(t-\tau)g(\tau)\,d\tau} .$$

The boxed formula is the way that Boyce and DiPrima define the convolution product $f*g$; see (2) on p. 346. This formula is valid when f and g are merely piecewise continuous functions on every bounded interval $[0, A]$.

Ex 1 (Similar to #3, p. 350) Compute the convolution product of $f(t)=t$ and $g(t)=t^2$.

<u>Solution:</u>

$$
\begin{aligned}
(f*g)(t) &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t (t-\tau)\tau^2\,d\tau \\
&= \int_0^t (t\tau^2-\tau^3)\,d\tau \\
&= \left.\frac{t\tau^3}{3}-\frac{\tau^4}{4}\right|_{\tau=0}^{t} \\
&= \boxed{\frac{t^4}{12}}.
\end{aligned}
$$

<u>Challenge:</u> Show that $t^n * t^m = \dfrac{m!\,n!\,t^{m+n+1}}{(m+n+1)!}$ for nonnegative integers m and n.

<u>The Convolution Theorem</u> (6.6.1 on pp. 345-6) If f and g are piecewise continuous functions that are of exponential order on $[0,\infty)$ then

$$\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f\}(s)\,\mathcal{L}\{g\}(s).$$

(See pp. 347-8 for a proof.)

Ex 2 (Similar to #4, #5 on p. 350) Use the convolution theorem to compute $\mathcal{L}\{f*g\}(s)$ if $f(t)=t$ and $g(t)=t^2$. Then check your work by using the result of example 1.

Solution: By the convolution theorem,

$$\begin{aligned}\mathcal{L}\{f*g\}(s) &= \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(s)\\ &= \mathcal{L}\{t\}(s)\mathcal{L}\{t^2\}(s)\\ &= \frac{1}{s^2}\cdot\frac{2}{s^3} = \boxed{\frac{2}{s^5}}.\end{aligned}$$

Same.

Using the result of Ex 1,

$$\mathcal{L}\{f*g\}(s) = \mathcal{L}\left\{\frac{t^4}{12}\right\}(s) = \frac{1}{12}\cdot\frac{4!}{s^5} = \boxed{\frac{2}{s^5}}$$

Ex 3 (Similar to #4, #5 on p. 350) Use the convolution theorem to compute $\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\}(s)$.

Solution: From the definition of the convolution product,

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau,$$

we see that $\int_0^t \tau e^{t-\tau}\,d\tau$ is the convolution product of $f(t) = e^t$ and $g(t) = t$. By the convolution theorem,

$$\begin{aligned}\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\}(s) &= \mathcal{L}\{e^t\}(s)\mathcal{L}\{t\}(s)\\ &= \frac{1}{s-1}\cdot\frac{1}{s^2}\\ &= \boxed{\frac{1}{s^2(s-1)}}.\end{aligned}$$

Ex 4 (Final Exam, Fall 2006; similar to #21, p. 351) Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)\,d\tau = 1$.

Note: Such an equation is called "integral" because the unknown function y appears inside an integral in the equation.

Solution: We use Laplace transforms and convolutions. Observe first that $\int_0^t (t-\tau)y(\tau)\,d\tau$ is the convolution of $f(t) = t$ with $y(t)$.

Therefore, we are asked to solve

$$y(t) + t * y(t) = 1.$$

Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{y\}(s) + \mathcal{L}\{t * y(t)\}(s) = \mathcal{L}\{1\}(s)$$

$$\mathcal{L}\{y\}(s) + \mathcal{L}\{t\}(s)\,\mathcal{L}\{y\}(s) = \frac{1}{s}$$

$$\mathcal{L}\{y\}(s) + \frac{1}{s^2}\mathcal{L}\{y\}(s) = \frac{1}{s}$$

$$s^2\mathcal{L}\{y\}(s) + \mathcal{L}\{y\}(s) = s$$

$$(s^2+1)\mathcal{L}\{y\}(s) = s$$

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+1}.$$

Therefore $y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \boxed{\cos(t)}$

<u>Ex 5</u> (Similar to #15, 17, 19 on p.351) Find a formula for the solution to the IVP

$$(*) \qquad y''' + 4y' = f(t), \qquad y(0) = 1,\ y'(0) = 0 = y''(0),$$

in terms of the convolution product. Assume that f is piecewise continuous and of exponential order on $[0, \infty)$, but otherwise is an arbitrary function.

<u>Solution</u>: We use the Laplace transform method. We begin by taking the Laplace transform of both sides of the differential equation in (*):

$$\mathcal{L}\{y''' + 4y'\}(s) = \mathcal{L}\{f(t)\}(s) = F(s)$$

$$s^3\mathcal{L}\{y\}(s) - s^2\overset{1}{\cancel{y(0)}} - s\overset{0}{\cancel{y'(0)}} - \overset{0}{\cancel{y''(0)}} + 4\left(s\mathcal{L}\{y\}(s) - \overset{1}{\cancel{y(0)}}\right) = F(s)$$

$$(s^3 + 4s)\mathcal{L}\{y\}(s) = s^2 + 4 + F(s)$$

$$\mathcal{L}\{y\}(s) = \frac{\overset{1}{\cancel{s^2+4}}}{s\cancel{(s^2+4)}} + \frac{F(s)}{s(s^2+4)}.$$

Taking the inverse Laplace transform we obtain

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s(s^2+4)}\right\}.$$

<u>Note</u>: The function $H(s) = \dfrac{1}{s(s^2+4)}$ is called the <u>transfer function</u> of the input-output system described by (*).

Therefore,

$$y(t) = 1 + f * h(t)$$

by the convolution theorem (pp. 345-6); here

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} \underset{\substack{\uparrow \\ \text{routine} \\ \text{P.F.D.}}}{=} \mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{\frac{1}{4}s}{s^2+4}\right\} = \frac{1}{4} - \frac{1}{4}\cos(2t).$$

Consequently, using the definition of the convolution product,

$$f*h(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau,$$

we have

$$\boxed{y(t) = 1 + \frac{1}{4}\int_0^t f(\tau)\left[1 - \cos(2(t-\tau))\right]d\tau}.$$

Notes: This formula allows us to express the "output" $y(t)$ of the input-output system $(*)$ in terms of the "input" $f(t)$. Such formulas are used in control problems to produce a desired output by adjusting the input $f(t)$ appropriately.

Challenge Problem: Find a formula in terms of the convolution product for the solution to the initial value problem

$$(**) \qquad y'' + by' + cy = f(t), \qquad y(0) = y_0, \; y'(0) = y_1,$$

where b, c, y_0, and y_1 are arbitrary constants with $4c > b^2$ and where the input $f(t)$ is piecewise continuous and of exponential order on $[0,\infty)$. What is the transfer function $H(s)$ of the system modeled by $(**)$?