Sec. 7.3: Systems of Algebraic Equations; Linear Independence; Eigenvalues and Eigenvectors

HW p.383: #3, 7, 16, 17   Due: Fri., Nov. 11

Schaum's: p.133. Also #15.12–15.14 and #15.32–15.40.

Please read the textbook pp. 373–376 regarding systems of (linear) algebraic equations and their solutions. In particular, note the method of Gaussian elimination illustrated in examples 1 and 2.

Linear Independence (p.377)

A set of k column vectors $\vec{x}^{(1)}, \dots, \vec{x}^{(k)}$ is called <u>linearly dependent</u> if there exist k (possibly complex) constants $c_1, \dots, c_k$, at least one of which is nonzero, such that

$$(*) \qquad c_1\vec{x}^{(1)} + \dots + c_k\vec{x}^{(k)} = \vec{0}.$$

If the only set of k constants $c_1, \dots, c_k$ for which $(*)$ holds is the trivial set $0 = c_1 = \dots = c_k$, then we call $\vec{x}^{(1)}, \dots, \vec{x}^{(k)}$ a <u>linearly independent</u> set.

<u>Ex1</u> (#8, p. 383) Determine whether

$$\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}^{(3)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

is a linearly independent set of vectors. If they are linearly dependent, find a nontrivial linear relation (like $(*)$ above) among them.

<u>Solution</u>: We want to know if the only choice of constants c_1, c_2, c_3 such that

$$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + c_3\vec{x}^{(3)} = \vec{0}$$

is the trivial choice: $c_1 = c_2 = c_3 = 0$. That is,

$$(*) \qquad c_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

But this vector equation is equivalent to the vector-matrix equation

$$(**)\qquad \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and according to the discussion on p. 374 of the textbook, this vector-matrix equation has a nonzero solution $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ if and only if

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}) = \det\left[\vec{x}^{(1)}\ \vec{x}^{(2)}\ \vec{x}^{(3)}\right] = \det\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

is equal to zero. But clearly $\det\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 0$ by expansion by cofactors along the third row (which is the zero vector). Therefore $\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}^{(3)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ form a $\boxed{\text{linearly dependent}}$ set of vectors.

To find a nontrivial linear relation (*) between these three vectors, we solve the system (**) using Gaussian elimination. We write the augmented matrix for (**) and perform elementary row operations to place it in "reduced echelon" form.

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-2)\cdot R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 0 & \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)\cdot R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ The equivalent system is $$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

whose solution is $c_1 = \frac{1}{2}c_3$, $c_2 = -\frac{5}{2}c_3$ with c_3 arbitrary:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_3 \\ -\frac{5}{2}c_3 \\ c_3 \end{bmatrix} = c_3\begin{bmatrix} \frac{1}{2} \\ -\frac{5}{2} \\ 1 \end{bmatrix}.$$

We can choose any (nonzero) value for c_3 to obtain a nontrivial linear relation among $\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}$. The choice $c_3 = 2$ leads to "simple" solution:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 2 \begin{bmatrix} 1/2 \\ -5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}.$$

One observes that

$$\boxed{1 \vec{x}^{(1)} - 5\vec{x}^{(2)} + 2\vec{x}^{(3)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}}$$

is a nontrivial linear relation among the vectors $\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}$.

<u>Eigenvalues/Eigenvectors of a (square) Matrix</u> (p. 379)

The number λ is an <u>eigenvalue</u> of the square matrix A provided there exists a nonzero column vector $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$. In this case the vector $\vec{x}$ is called an <u>eigenvector</u> of A corresponding to λ.

For example, consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Note that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore $\lambda_1 = 2$ and $\lambda_2 = 0$ are eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with corresponding eigenvectors $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

Q1: Does $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ have any more eigenvalues?

Q2: How can one compute the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$?

These questions are answered by the following result (see p. 379).

FACT: The eigenvalues of the $n \times n$ matrix A are precisely the solutions to the n^{th} degree polynomial equation in λ:

$$(\square) \qquad \det(A - \lambda I) = 0.$$

Notes: The equation in $(\square)$ is called the characteristic equation of A. Since the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is 2×2, its characteristic equation is a quadratic equation in λ so there are only 2 eigenvalues for A.

Ex 2| Use the characteristic equation to compute the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then find the corresponding eigenvectors of A.

Solution: The characteristic equation of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$$
$$= \lambda(\lambda - 2)$$

Therefore the eigenvalues of A are $\boxed{\lambda_1 = 2 \text{ and } \lambda_2 = 0}$. The corresponding eigenvectors $\vec{x}$ solve $A\vec{x} = \lambda\vec{x}$ or equivalently $A\vec{x} - \lambda\vec{x} = \vec{0}$, which is again equivalent to $(A-\lambda I)\vec{x}=\vec{0}$ or $\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (†)

$\underline{\lambda_1 = 2}$: We need to solve (†) with $\lambda = 2$. Thus

$$\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{cases} -x_1 + x_2 = 0 \\ \cancel{x_1 - x_2 = 0}. \end{cases}$$ ← This equation is redundant. It is -1 times the first equation

Therefore $x_2 = x_1$ so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence an eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ corresponding to $\lambda_1 = 2$ is $\boxed{\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ (up to a constant factor).

$\underline{\lambda_2 = 0}$: We need to solve (†) with $\lambda = 0$. Consequently

$$\begin{bmatrix} 1-0 & 1 \\ 1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_1 + x_2 = 0 \\ \cancel{x_1 + x_2 = 0} \end{cases}$$ ← Redundant.

Therefore $x_2 = -x_1$ so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus an eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ corresponding to $\lambda_2 = 0$ is $\boxed{\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ (up to a constant factor).

Ex 4 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Solution: We use an HP-49G calculator to assist us. The calculator gives the following answers.

Eigenvalues of A	Corresponding Eigenvectors of A
$\lambda_1 = 8$	$\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$
$\lambda_2 = -1$	$\vec{x}^{(2)} = \begin{bmatrix} 36 \\ 0 \\ -36 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
$\lambda_3 = -1$	$\vec{x}^{(3)} = \begin{bmatrix} 0 \\ 36 \\ -18 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

We note that when a matrix has repeated eigenvalues the corresponding eigenvectors are not unique. For example, a hand calculation (see the following pages) shows that

$$\vec{y}^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{y}^{(3)} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

is another linearly independent pair of eigenvectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ corresponding to $\lambda = -1$.

Appendix: Hand computation of the eigenvalues and eigenvectors of $A=\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Expansion by cofactors along row 2

$$0=\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (-2)\begin{vmatrix} 2 & 4 \\ 2 & 3-\lambda \end{vmatrix} + (-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 3-\lambda & 2 \\ 4 & 2 \end{vmatrix}$$

$$0=-2\left(2(3-\lambda)-8\right)-\lambda\left((3-\lambda)^2-16\right)-2\left(2(3-\lambda)-8\right)$$

$$0= -\lambda^3+6\lambda^2+15\lambda+8$$

The candidates for rational zeros are integers that divide 8; namely $\pm1, \pm2, \pm4, \pm8$. By inspection, it is easy to see that $\lambda=-1$ is a solution. Consequently $\lambda+1$ is a factor of the cubic polynomial in λ, and we use synthetic division and deflation to obtain a complete factorization.

$$\begin{array}{r|rrrr}
-1 & -1 & 6 & 15 & 8 \\
 & & 1 & -7 & -8 \\
\hline
-1 & -1 & 7 & 8 & 0 \\
 & & 1 & -8 & \\
\hline
 & -1 & 8 & 0 &
\end{array}$$

Therefore $0=(\lambda+1)(\lambda+1)(-\lambda+8)$ so the eigenvalues of A are

$\boxed{\lambda=-1 \text{ (multiplicity two) and } \lambda=8.}$

The eigenvectors $\vec{x}$ of A satisfy $(A-\lambda I)\vec{x}=\vec{0}$ or $\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\underline{\lambda=8}$: $\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ We use Gaussian elimination.

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ -5 & 2 & 4 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{5R_1+R_2} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{9}R_2]{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} x_1 - x_3 = 0 \\ 2x_2 - x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = x_3 \\ x_2 = \frac{1}{2}x_3 \end{array}$

Thus $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$. We take $x_3 = 2$ to get a "clean"

eigenvector $\boxed{\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$ of A corresponding to $\lambda = 8$.

$\lambda = -1$: $\begin{bmatrix} 3-(-1) & 2 & 4 \\ 2 & -(-1) & 2 \\ 4 & 2 & 3-(-1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \xrightarrow[-1R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ Therefore

$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{cases} 2x_1 + x_2 + 2x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$ which implies $x_2 = -2x_1 - 2x_3$.

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 - 2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ where x_1 and x_3 are free variables.

Taking $x_1 = 1, x_3 = 0$ gives $\boxed{\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}$. Taking $x_1 = 0, x_3 = 1$ gives $\boxed{\vec{x}^{(3)} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}$. These are two linearly independent eigenvectors of A corresponding to $\lambda = -1$.