Sec. 7.4: Basic Theory of Systems of First Order Linear Equations

HW p. 389: # 1, 2, 3, 6 Due: Mon., Nov. 15

Schaum's: ??

Consider a system of 2 first order linear DE's:

$$(*)\quad \begin{cases} x_1' = p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + g_1(t) \\ x_2' = p_{21}(t)\,x_1 + p_{22}(t)\,x_2 + g_2(t). \end{cases}$$

An equivalent vector-matrix formulation of $(*)$ is

$$(**)\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}.$$

Ex 1 (#22, pp. 362-3) The two-tank flow process was modeled by

$$\begin{cases} Q_1' = -0.1\,Q_1 + 0.075\,Q_2 + 1.5 \\ Q_2' = 0.1\,Q_1 - 0.2\,Q_2 + 3 \end{cases}$$

Write an equivalent vector-matrix formulation of this system.

Solution:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}' = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}.$$

In general, a system of n first order linear DE's can be expressed in vector-matrix form as

$$(2)\qquad \vec{x}\,' = P(t)\,\vec{x} + \vec{g}(t)$$

where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{g}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$ are n-vector functions

and
$$P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

is an $n\times n$ matrix function. If $\vec{g}(t) = \vec{0}$ in (2) then the system is called homogeneous:

$$(3) \qquad \vec{x}\,' = P(t)\vec{x}.$$

In Secs. 7.4 – 7.8 we will be concerned with solving homogeneous linear systems. The following Superposition Principle is a fundamental tool in solving such systems.

Theorem 7.4.1 (p. 386): If $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are solutions of the homogeneous system (3) then so is the linear combination $c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$ for any constants c_1 and c_2.

Q: What is the most general solution of (3)?

A: Theorem 7.4.2 (p. 387): If $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$ are n solutions of the system of n first order linear homogeneous DE's

$$(3) \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

on an interval $\alpha < t < \beta$ and $W(\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)) \neq 0$ for all

$\alpha < t < \beta$, then every solution of (3) on $\alpha < t < \beta$ has the form

$$(\dagger) \qquad \vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t) \qquad (\alpha < t < \beta)$$

for some constants $c_1, \dots, c_n$.

<u>Notes</u>: We say that $(\dagger)$ is the <u>general solution</u> of (3) in the interval $\alpha < t < \beta$ and the collection $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ is called a <u>fundamental</u> <u>set</u> <u>of</u> <u>solutions</u> of (3) in $\alpha < t < \beta$.

A key tool for analyzing the Wronskian is Abel's identity:

$$\frac{dW}{dt} = \left(p_{11}(t) + p_{22}(t) + \dots + p_{nn}(t) \right) W,$$

satisfied by the Wronskian $W(\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t))$ of <u>any</u> set of n solutions $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ to (3). Equivalently, in integrated form the Wronskian satisfies

$$W\left(\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)\right) = c \exp\left\{ \int \left[p_{11}(t) + p_{22}(t) + \dots + p_{nn}(t) \right] dt \right\}$$

for some constant c and all $\alpha < t < \beta$. The next homework problem outlines a proof of Abel's identity when $n = 2$.

<u>Ex 2</u> (#2, p. 389) Let $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ be solutions of

$$(3) \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and let W be the Wronskian of $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$.

(a) Show that $\frac{dW}{dt} = \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} \end{vmatrix}$.

(b) Using equation (3), show that $\frac{dW}{dt} = (p_{11} + p_{22})W$.

(c) Find $W(t)$ by solving the differential relation obtained in part (b).

Solution : (a) $W(t) = \begin{vmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{vmatrix} = x_1^{(1)}(t)\, x_2^{(2)}(t) - x_2^{(1)}(t)\, x_1^{(2)}(t)$

$$\text{so } \frac{dW}{dt} = \frac{dx_1^{(1)}(t)}{dt} x_2^{(2)}(t) + x_1^{(1)}(t) \frac{dx_2^{(2)}(t)}{dt} - \frac{dx_2^{(1)}(t)}{dt} x_1^{(2)}(t) - x_2^{(1)}(t) \frac{dx_1^{(2)}(t)}{dt}$$

$$= \left(\frac{dx_1^{(1)}(t)}{dt} x_2^{(2)}(t) - x_2^{(1)}(t) \frac{dx_1^{(2)}(t)}{dt} \right) + \left(x_1^{(1)}(t) \frac{dx_2^{(2)}(t)}{dt} - \frac{dx_2^{(1)}(t)}{dt} x_1^{(2)}(t) \right)$$

$$\overset{\checkmark}{=} \begin{vmatrix} \frac{dx_1^{(1)}(t)}{dt} & \frac{dx_1^{(2)}(t)}{dt} \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{vmatrix} + \begin{vmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ \frac{dx_2^{(1)}(t)}{dt} & \frac{dx_2^{(2)}(t)}{dt} \end{vmatrix}.$$

(b) From part (a) and equation (3),

$$(\blacksquare) \qquad \frac{dW}{dt} = \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} \end{vmatrix}$$

$$= \begin{vmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} & p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} & p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{vmatrix}$$

But

Add $-p_{12}$ times the second row to the first row

$$\begin{vmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} & p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = \begin{vmatrix} p_{11}x_1^{(1)} & p_{11}x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$= p_{11}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$= p_{11}W \qquad (\blacksquare\blacksquare)$$

and

Add $-p_{21}$ times the first row to the second row

$$\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} & p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{vmatrix} = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{22}x_2^{(1)} & p_{22}x_2^{(2)} \end{vmatrix}$$

$$= p_{22}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$= p_{22}W. \qquad (\blacksquare\blacksquare\blacksquare)$$

Substituting from $(\blacksquare\blacksquare)$ and $(\blacksquare\blacksquare\blacksquare)$ into $(\blacksquare)$ gives the desired result:

$$\frac{dW}{dt} = p_{11}W + p_{22}W = (p_{11} + p_{22})W.$$

(c) Separating variables in the differential equation from part (b) yields

$$\frac{dW}{W} = (p_{11} + p_{22})dt .$$

Integrating both sides we have

$$\ln|W| = \int_{t_0}^{t} [p_{11}(s) + p_{22}(s)]ds + c_1$$

and $$W(t) = c \exp\left\{ \int_{t_0}^{t} [p_{11}(s) + p_{22}(s)]ds \right\}$$

where $c = \pm e^{c_1}$ and t_0 is a fixed point in the interval $\alpha < t < \beta$.

Ex 3] (#3, p.389) Show that the Wronskians of two fundamental sets of solutions of the system

$$(3) \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

on an interval $\alpha < t < \beta$ can differ at most by a multiplicative constant.

Solution: Let $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$ and $\vec{z}^{(1)}(t), \ldots, \vec{z}^{(n)}(t)$ be two fundamental sets of solutions to (3) on $\alpha < t < \beta$, and let

$$W_1(t) = \det\left[\vec{x}^{(1)}(t) \ \cdots \ \vec{x}^{(n)}(t) \right]$$

$$W_2(t) = \det\left[\vec{z}^{(1)}(t) \ \cdots \ \vec{z}^{(n)}(t) \right]$$

be their respective Wronskians. By problem 2 on p. 389, there exist constants c_1 and c_2 such that

$$W_1(t) = c_1 \exp\left\{ \int_{t_0}^{t} \left[p_{11}(s) + p_{22}(s) + \cdots + p_{nn}(s) \right] ds \right\}$$

and

$$W_2(t) = c_2 \exp\left\{ \int_{t_0}^{t} \left[p_{11}(s) + p_{22}(s) + \cdots + p_{nn}(s) \right] ds \right\}$$

for all $\alpha < t < \beta$. It is clear from these formulas that W_1 and W_2 differ at most by a multiplicative constant on $\alpha < t < \beta$.