Sec. 7.5: Homogeneous Linear Systems with Constant Coefficients

HW p. 398: # 1, 7, 20, 30 Due: Wed., Nov. 17

Schaum's:

In this and succeeding sections we take up the problem of finding the general solution of homogeneous linear systems

$$(*) \qquad \vec{x}' = A\vec{x}$$

where A is an $n \times n$ constant matrix. By analogy with the scalar problem $x' = ax$ which has general solution $x(t) = ke^{at}$, we expect $(*)$ to have exponential solutions: $\vec{x} = \vec{k}e^{\lambda t}$. Then $\vec{x}' = \vec{k}\lambda e^{\lambda t}$ so substituting in $(*)$ we would have

$$\vec{k}\lambda e^{\lambda t} = A\vec{k}e^{\lambda t},$$

and cancelling $e^{\lambda t}$ from both sides yields

$$\lambda\vec{k} = A\vec{k}.$$

This is the eigenvalue problem for the coefficient matrix A.

Ex 1 (#3, p. 398) (a) Find the general solution of the system

$$(*) \qquad \vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$$

and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field for $(*)$ and plot a few trajectories of the system.

Solution: (a) $\vec{x} = \vec{k}e^{\lambda t}$ in $(*)$ leads to $\lambda\vec{k} = A\vec{k}$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. We need to solve the eigenvalue problem for A.

$$0 = \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-2)+3 = \lambda^2-1 = (\lambda-1)(\lambda+1)$$

Eigenvalues	Eigenvectors
$\lambda_1 = 1$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -1$	$\vec{k}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(See work to right and below.)

$$(A-\lambda I)\vec{k} = \vec{0}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 1}$: $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left\{ \begin{array}{ll} k_1 - k_2 = 0 & k_1 = k_2 \\ \cancel{3k_1 - 3k_2 = 0} & \text{Redundant} \end{array} \right.$$

$$\therefore \vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\underline{\lambda = -1}$: $\begin{bmatrix} 3 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \Rightarrow \left\{ \begin{array}{l} 3k_1 - k_2 = 0 \\ \cancel{3k_1 - 3k_2 = 0} \ \text{Redundant} \end{array} \right. \Rightarrow k_2 = 3k_1$

$$\therefore \vec{k}^{(2)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 3k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Consequently $\vec{x}^{(1)} = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$ and $\vec{x}^{(2)} = \vec{k}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

are solutions.

$$W(\vec{x}^{(1)}\ \vec{x}^{(2)}) = \det\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{bmatrix} = 3e^{0} - e^{0} = 2 \neq 0,$$

so $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$, $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ form a F.S.S. for $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$.

The general solution is

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}}$$

where c_1 and c_2 are arbitrary constants. If $c_1 = 0$ then

$$\vec{x}(t) = c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} \longrightarrow \begin{bmatrix}0\\0\end{bmatrix} \text{ as } t\to\infty.$$

If $c_1 \neq 0$ then $\|\vec{x}(t)\| = \sqrt{(c_1e^{t}+c_2e^{-t})^2+(c_1e^{t}+3c_2e^{-t})^2} \longrightarrow \infty$ as $t\to+\infty$.

In fact, $\vec{x}(t) - c_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} = c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} \longrightarrow \begin{bmatrix}0\\0\end{bmatrix}$ as $t\to+\infty$.

(b) The vector-matrix system

$$(*) \qquad \begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

is equivalent to the system of 2 coupled scalar DE's:

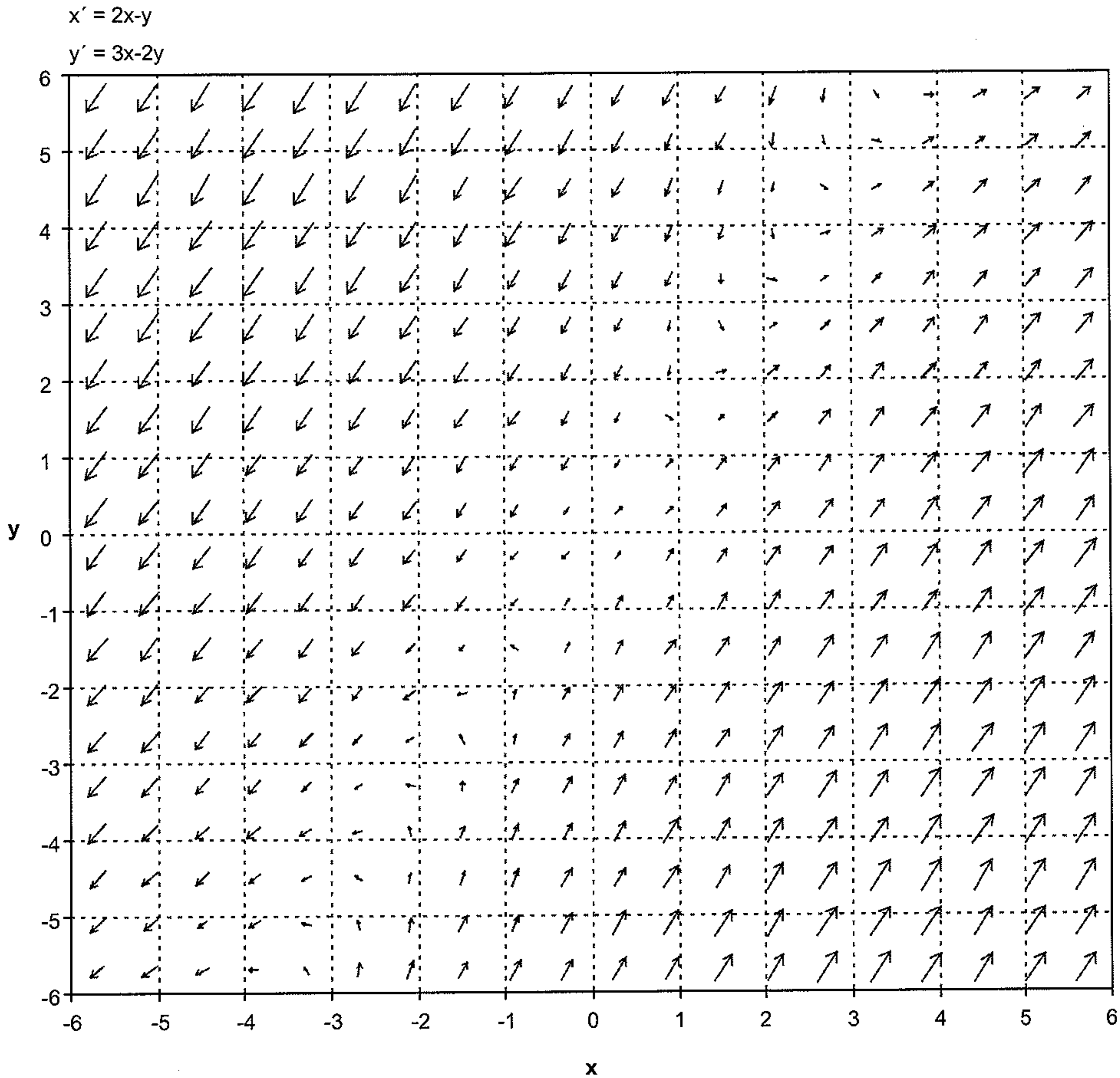
$$(**) \qquad \begin{cases} \dfrac{dx}{dt} = 2x - y \\[2mm] \dfrac{dy}{dt} = 3x - 2y \end{cases}.$$

We can eliminate the parameter t using the chain rule $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ and $(**)$ to obtain

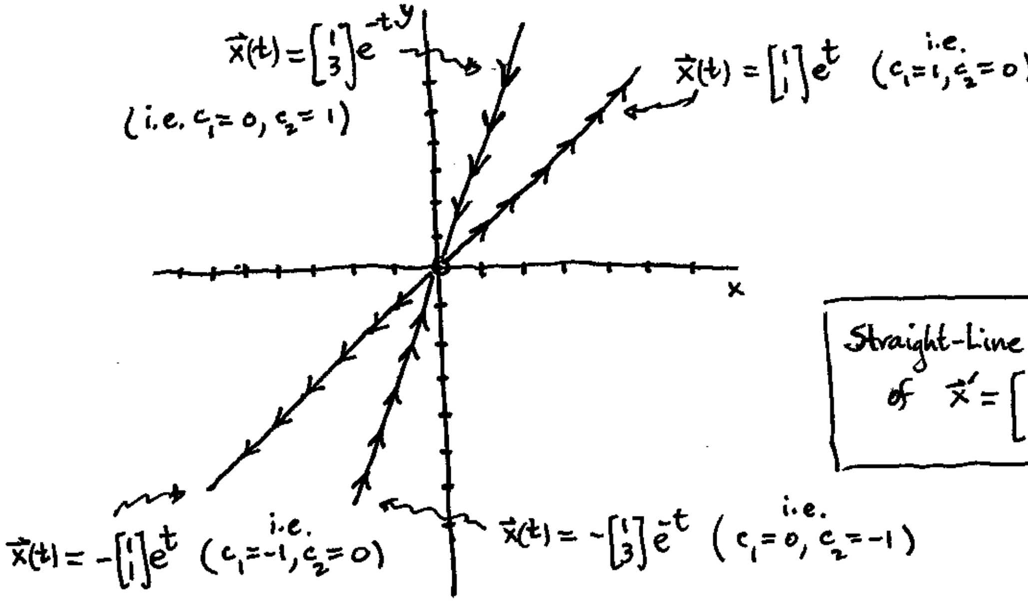
$$(***) \qquad \frac{dy}{dx} = \frac{3x-2y}{2x-y}.$$

Therefore the <u>direction field</u> of $(*)$ is the same as the direction field of $(***)$, and the latter can be obtained using the techniques we learned in Chapter 1.

See the next page for a print out of the direction field of $(*)$.


x′ = 2x-y
y′ = 3x-2y
y
x
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6

A trajectory of the system $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$ results when we make a specific choice for the constants c_1 and c_2 in the general solution $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$. It is easy to sketch the "straight-line" trajectories resulting from $c_1 = \pm 1, c_2 = 0$ and from $c_1 = 0, c_2 = \pm 1$. The results are shown below.

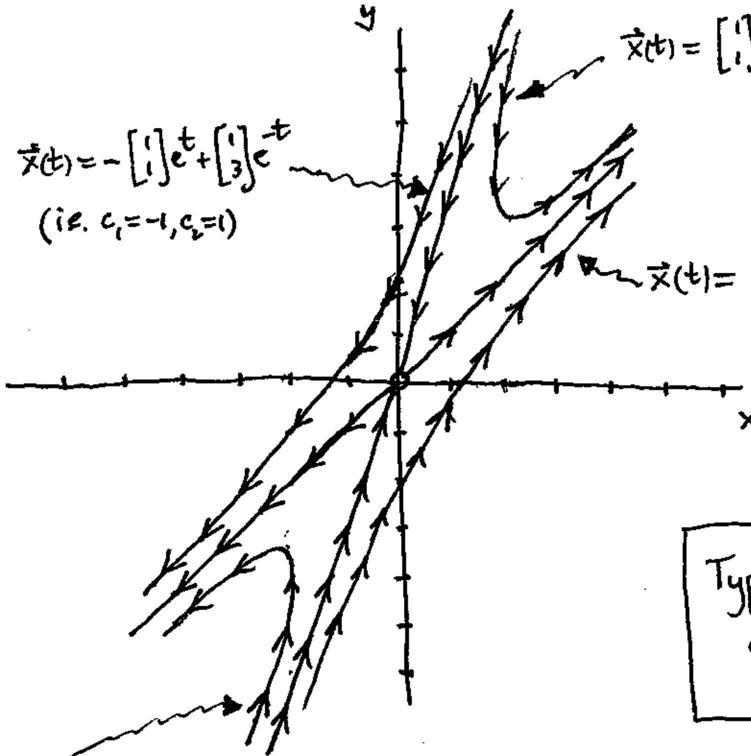


$$\boxed{\text{Straight-Line Trajectories of } \vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}}$$

Notice in particular that arrows are used to indicate the "motion" of the particle governed by the vector function $\vec{x}(t)$ as t increases. To obtain trajectories in the "cones" between the straight-line trajectories above, we rewrite the general solution $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ as scalar parametric equations:

$$\begin{cases} x(t) = c_1 e^{t} + c_2 e^{-t} \\ y(t) = c_1 e^{t} + 3c_2 e^{-t} \end{cases}$$

Using a graphing calculator (in parametric mode) we can obtain four by choosing $c_1 = \pm 1, c_2 = \pm 1$. The results are shown on the next page.



Typical Trajectories of $\vec{x}' = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\vec{x}$

Math 204 Supplement 2 to Section 7.5
Direction Fields for Systems

Section 7.5 of Boyce and DiPrima uses direction fields as an aid in analyzing the qualitative behavior of solutions to homogeneous linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a constant matrix. However, direction fields are displayed in Figures 7.5.1 and 7.5.3 with no indication as to how they were generated. The purpose of this supplement is to indicate how one can easily generate such direction fields with the aid of a graphing calculator and the techniques of Chapter 1.

Consider the homogeneous system

$$(*) \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

where $a, b, c,$ and d are constants. This vector-matrix formulation of the system is equivalent to the following coupled system of scalar differential equations.

$$(**) \qquad \begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

The parameter t can be eliminated in this system using the chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

and the equations (**) to produce a first order scalar differential equation equivalent to (*):

$$(***) \qquad \frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

The direction field of (*) can thus be obtained from the direction field of (***) using the techniques that were employed in Chapter 1 for first order scalar differential equations.

Example 1 (p. 391). Plot a direction field for the system $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}$.

Solution: Using (***) we have the equivalent (scalar) first order differential equation

$$(+) \qquad \frac{dy}{dx} = \frac{4x + y}{x + y}.$$

Using a graphical calculator to generate the slopefield (i.e. direction field) of (+) as in Chapter 1, we produce Figure 7.5.1.

Example 2 (p. 394). Plot a direction field for the system $\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\mathbf{x}$.

Solution: Using (***) we have the equivalent (scalar) first order differential equation

$$(++) \qquad \frac{dy}{dx} = \frac{\sqrt{2}x - 2y}{-3x + \sqrt{2}y}.$$

As we did in Chapter 1, we use a graphical calculator to generate the slopefield of (++) and thus produce Figure 7.5.3.

Subject: Math 204 - phase plane tool online
From: "Singler, John" <singlerj@mst.edu>
To: "Grow, David E." <grow@mst.edu>,
...snip... "Wintz, Nicholas J." <njwn7d@mst.edu>
Cc: <spaceman@fidmail.com>

Hello everyone,
Here is the website that has a fairly easy to use tool for plotting direction fields/phase planes for linear DE systems:

http://math.rice.edu/~dfield/dfpp.html

The pplane java applet is easy to use. I have used this tool in my mathematical modeling class with good results. You can go to Gallery -> Linear System to enter the DE, and then on the plot you can click to show solution trajectories. This, hopefully, would make it fairly straightforward for the students. Also, it would give us a chance to show them some nonlinear differential equations with very interesting solution behavior (the gallery has a lot of nice examples).
John

John R. Singler

Assistant Professor
Department of Mathematics and Statistics
Missouri University of Science and Technology
(Missouri S&T, formerly University of Missouri-Rolla)
400 W. 12th St.
Rolla, MO 65409-0020

Office: 312 Rolla Building
Phone: (573) 341-4648
Email: singlerj@mst.edu
URL: http://web.mst.edu/~singlerj/