Sec. 7.6: Complex Eigenvalues

HW p. 409: #1, 9, 25, 29 Due: Fri, Nov. 19

Schaum's:

Ex 1 (Similar to #9, p. 410) Find the solution of the initial value problem

$$\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Describe the behavior of the solution as $t \to \infty$. Sketch the trajectory in the x_1x_2-plane and draw the direction field of the system.

Solution: Let $A = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}$. Then $\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -8 \\ 1 & -3-\lambda \end{vmatrix} = (3+\lambda)(\lambda-1) + 8 = \lambda^2 + 2\lambda + 5.$$

$$\therefore \quad \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

An eigenvector of A corresponding to $\lambda = -1+2i$ satisfies $(A - \lambda I)\vec{k} = \vec{0}$,

$$\begin{bmatrix} 1-(-1+2i) & -8 \\ 1 & -3-(-1+2i) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{cases} \cancel{(2-2i)k_1 - 8k_2 = 0} \\ k_1 + (-2-2i)k_2 = 0 \end{cases}$$

Redundant. Equation 1 is (2-2i) times Equation 2.

Therefore $k_1 = (2+2i)k_2$ so

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} (2+2i)k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$$ is an eigenvector of A corresponding to $\lambda = -1+2i$.

Hence $\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{\lambda_1 t} = \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}e^{(-1+2i)t}$ is a complex vector solution to $\vec{x}' = A\vec{x}$. It is easy to verify that $\vec{x}^{(2)}(t) = \overline{\vec{k}^{(1)} e^{\lambda_1 t}} = \begin{bmatrix} 2-2i \\ 1 \end{bmatrix} e^{(-1-2i)t}$

is a complex vector solution to $\vec{x}' = A\vec{x}$ also. In order to get <u>real</u> vector solutions we use the superposition principle to see that

$$\tilde{\vec{x}}^{(1)} = \operatorname{Re}\left(\vec{x}^{(1)}(t)\right) = \frac{1}{2}\vec{x}^{(1)}(t) + \frac{1}{2}\overline{\vec{x}^{(1)}(t)} = \frac{1}{2}\vec{x}^{(1)}(t) + \frac{1}{2}\vec{x}^{(2)}(t)$$

$$\tilde{\vec{x}}^{(2)} = \operatorname{Im}\left(\vec{x}^{(1)}(t)\right) = \frac{1}{2i}\vec{x}^{(1)}(t) - \frac{1}{2i}\overline{\vec{x}^{(1)}(t)} = \frac{1}{2i}\vec{x}^{(1)}(t) - \frac{1}{2i}\vec{x}^{(2)}(t)$$

linear combinations of solutions to $\vec{x}' = A\vec{x}$ are again solutions.

are both (real) vector solutions to $\vec{x}' = A\vec{x}$.

<u>Monkey Face</u>: "Eyebrows" give real part; "chin" and "mouth" give imaginary part.

Note that

$$\vec{x}^{(1)}(t) = \vec{k}^{(1)} e^{\lambda t} = e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)\left(\cos(2t) + i\sin(2t)\right)$$

so

$$\tilde{\vec{x}}^{(1)} = \operatorname{Re}\left(\vec{x}^{(1)}(t)\right) = e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) = e^{-t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix}$$

$$\tilde{\vec{x}}^{(2)} = \operatorname{Im}\left(\vec{x}^{(1)}(t)\right) = e^{-t}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(2t)\right) = e^{-t}\begin{bmatrix} 2\cos(2t) + 2\sin(2t) \\ \sin(2t) \end{bmatrix}$$

is a (real-valued) F.S.S. because $W\left(\tilde{\vec{x}}^{(1)}(0), \tilde{\vec{x}}^{(2)}(0)\right) = \det\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0$.

The general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos(2t) + 2\sin(2t) \\ \sin(2t) \end{bmatrix}$$

where c_1 and c_2 are arbitrary constants. We need to choose the constants so

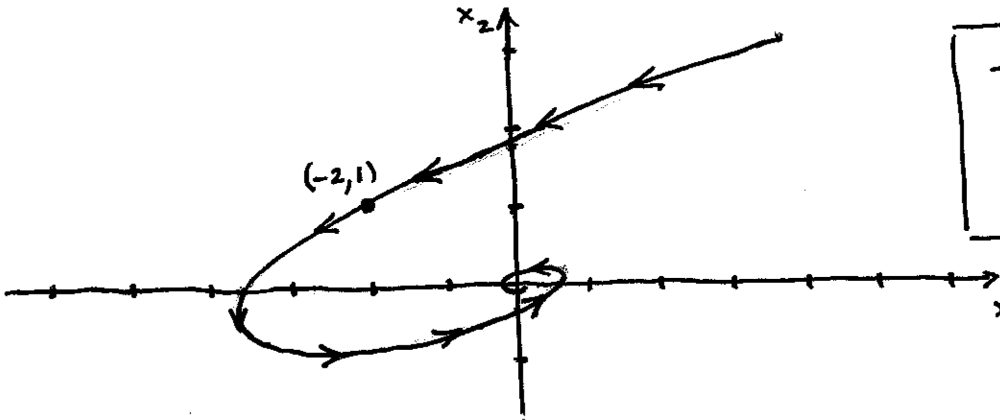
$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so $c_1 = 1$ and $c_2 = -2$. Thus

$$\vec{x}(t) = e^{-t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} - 2e^{-t}\begin{bmatrix} 2\cos(2t) + 2\sin(2t) \\ \sin(2t) \end{bmatrix} = \boxed{e^{-t}\begin{bmatrix} -2\cos(2t) - 6\sin(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix}}$$

solves the IVP. Since $\begin{bmatrix} -2\cos(2t) - 6\sin(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix}$ is a bounded vector function and $e^{-t} \to 0$ as $t \to \infty$, the solution $\vec{x}(t) \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

To sketch the trajectory in the x_1x_2-plane, we write the solution in parametric form and use a calculator:

$$\begin{cases} x_1(t) = e^{-t}\left(-2\cos(2t) - 6\sin(2t)\right) \\ x_2(t) = e^{-t}\left(\cos(2t) - 2\sin(2t)\right). \end{cases}$$

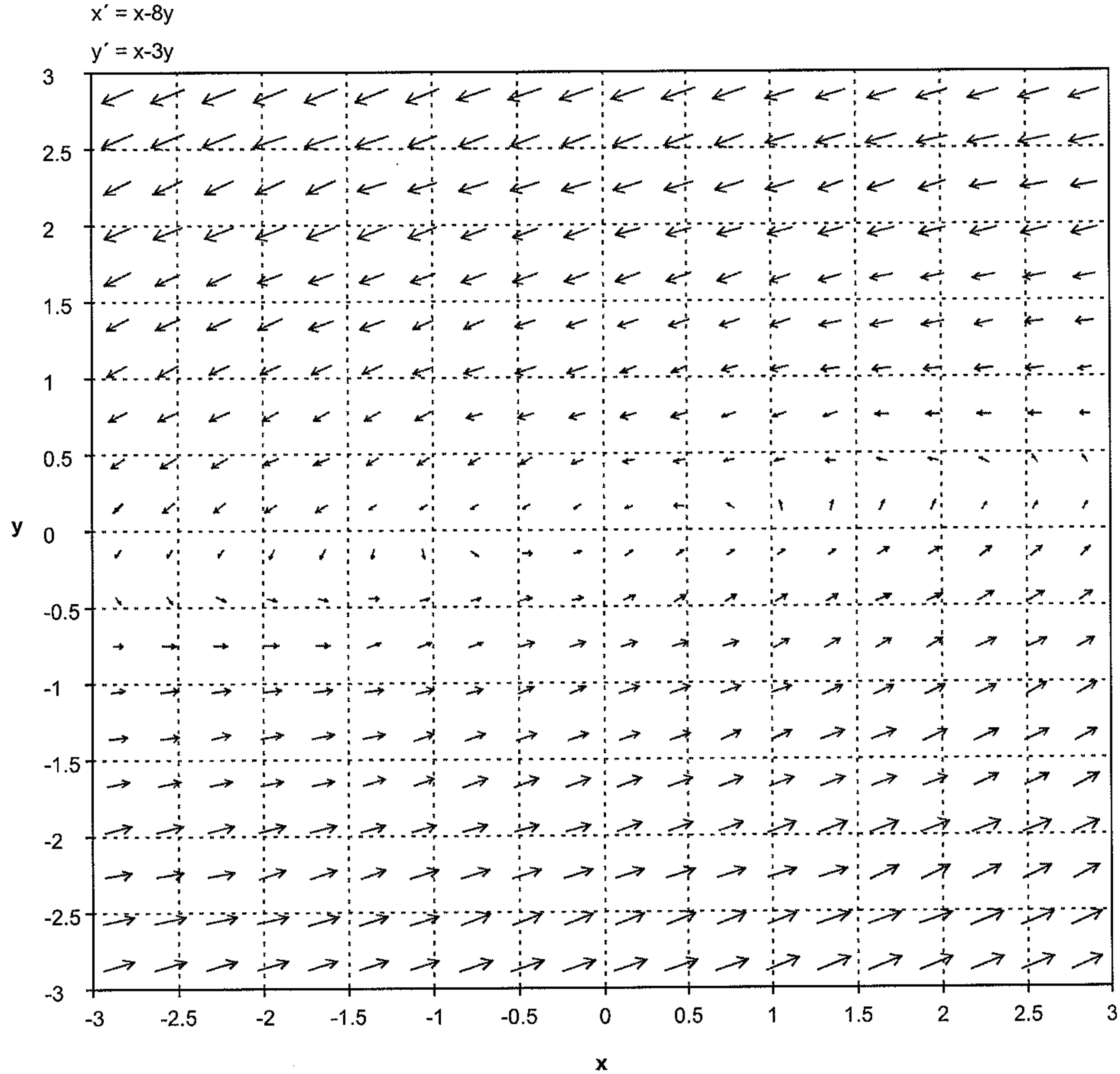


Trajectory of $\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}\vec{x}$ satisfying $\vec{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The direction field of $\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}\vec{x}$ is equivalent to that of

$$\frac{dx_2}{dx_1} = \frac{x_2'}{x_1'} = \frac{x_1 - 3x_2}{x_1 - 8x_2}$$ which can be obtained using a calculator

and the techniques of Chapter 1. The result is displayed on the next page.


x′ = x-8y
y′ = x-3y
y
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3