Sec. 7.8 Repeated Eigenvalues

HW p. 428: #1, 7, 13, 15   Due: Wed., Dec. 1

Ex 1 (Similar to #1, p. 428; Repeated Eigenvalues) Consider the system $\vec{x}' = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}\vec{x}$.

(a) Draw a direction field for the system.

(b) Find the general solution of the system and sketch a few trajectories in the x_1x_2-plane.

(c) Describe how solutions of the system behave as $t \to \infty$.

Solution: (a) We use the Rice University online tool to sketch a direction field for the system. (See the next page of these notes.)

(b) Let $A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$. Then $\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$. The eigenvalues of A are obtained via the characteristic equation:

$$0 = \det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix} = (\lambda+6)(\lambda-4) + 25 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2.$$

Eigenvalues	Eigenvectors
$\lambda_1 = -1$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -1$	$\vec{k}^{(2)}$ is not available.

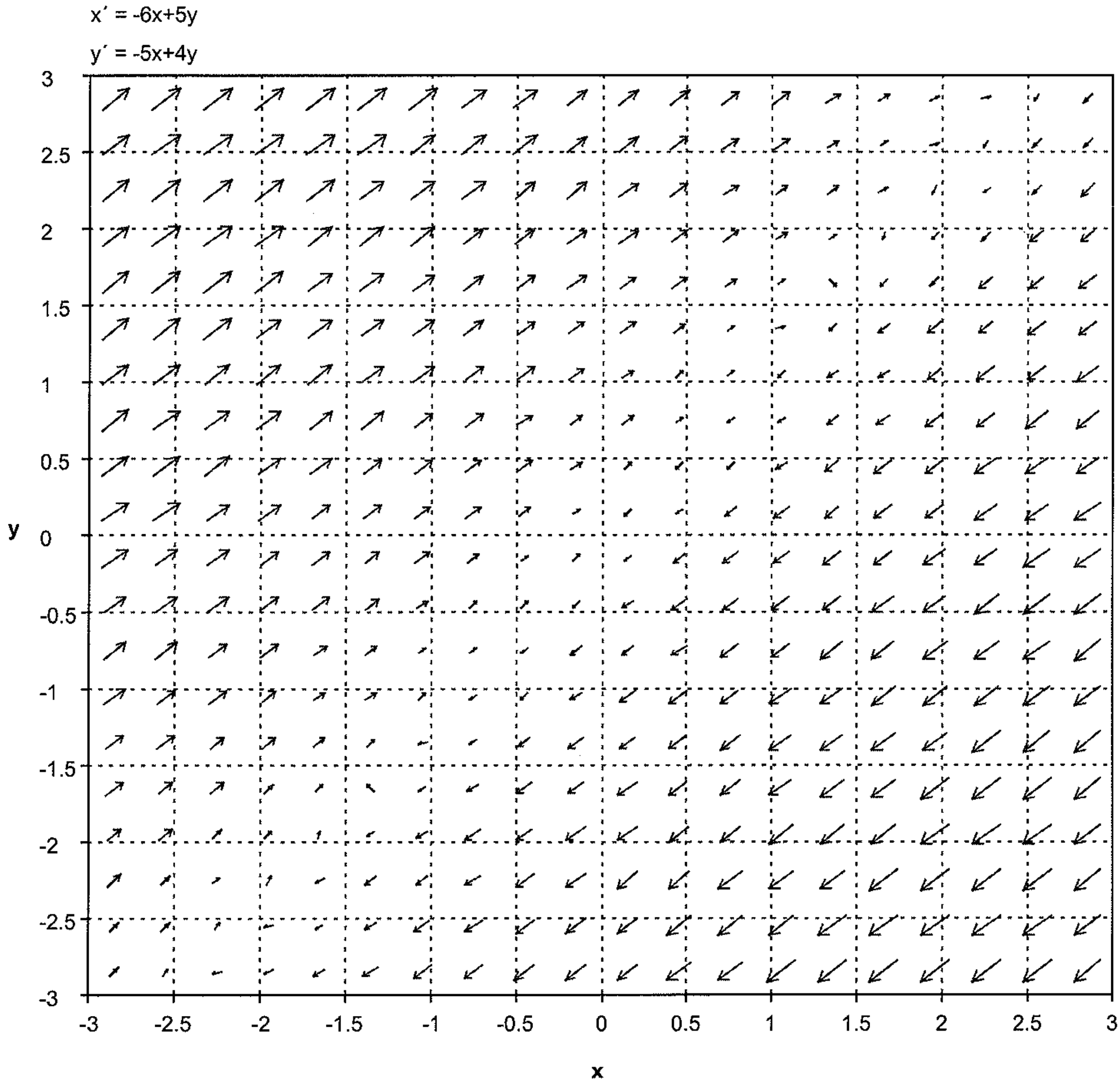
The eigenvectors of A satisfy $(A - \lambda I)\vec{k} = \vec{0}$. That is,

$$\begin{bmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ When $\lambda = -1$ this becomes

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so} \quad \begin{cases} -5k_1 + 5k_2 = 0 \Rightarrow k_2 = k_1 \\ \cancel{-5k_1 + 5k_2 = 0} \text{ Redundant.} \end{cases}$$

$\therefore$ an eigenvector of A corresponding to $\lambda = -1$ is $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We may take $k_1 = 1$ so $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note that the only eigenvectors of A corresponding to $\lambda = -1$ are constant multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore there is no second linearly independent eigenvector $\vec{k}^{(2)}$ corresponding to $\lambda_2 = -1$.

By analogy with the case of repeated roots to the characteristic equation for higher order linear homogeneous DEs with constant coefficients (see equation (26) on p. 170 and equation (18) on p. 230), we assume a solution to $\vec{x}' = A\vec{x}$ of the form $\vec{x}(t) = \vec{k}te^{\lambda t} + \vec{\ell}e^{\lambda t}$ when A has a


x′ = -6x+5y
y′ = -5x+4y
y
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

repeated eigenvalue and an insufficient number of corresponding (linearly independent) eigenvectors. Then

$$\vec{x}'(t) = \vec{k}e^{\lambda t} + \lambda\vec{k}te^{\lambda t} + \lambda\vec{l}e^{\lambda t}$$

so substituting in $\vec{x}' = A\vec{x}$ yields

$$(\vec{k} + \lambda\vec{k}t + \lambda\vec{l})e^{\lambda t} = A(\vec{k}t + \vec{l})e^{\lambda t}.$$

Cancelling $e^{\lambda t}$ and rearranging produces

$$\vec{0} = (A\vec{k} - \lambda\vec{k})t + (A\vec{l} - \lambda\vec{l} - \vec{k}).$$

Since this must hold for all t, it follows that

$$A\vec{k} - \lambda\vec{k} = \vec{0} \quad \text{and} \quad A\vec{l} - \lambda\vec{l} - \vec{k} = \vec{0}$$

That is,

$$\begin{cases} (A - \lambda I)\vec{k} = \vec{0} & (22) \\ \text{and} & \\ (A - \lambda I)\vec{l} = \vec{k} & (24) \end{cases}$$

Note that we have already solved (22) in the case $A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$ and found $\lambda_1 = -1$ and $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (up to a constant factor). Substituting for A, λ, and $\vec{k}$ in (24), we must solve

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\vec{l}$. But this is equivalent to asking that $-5l_1 + 5l_2 = 1$ so the solution is $l_2 = l_1 + \frac{1}{5}$ where l_1 is arbitrary. Thus

$$\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_1 + \frac{1}{5} \end{bmatrix} = l_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

We take $l_1 = 0$ for convenience, and hence $\vec{l} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$. Consequently

$$\vec{x}^{(2)}(t) = \vec{k}\,te^{\lambda t} + \vec{l}\,e^{\lambda t} = \begin{bmatrix}1\\1\end{bmatrix}te^{-t} + \begin{bmatrix}0\\1/5\end{bmatrix}e^{-t}$$ is another solution

to $\vec{x}' = \begin{bmatrix}-6 & 5\\-5 & 4\end{bmatrix}\vec{x}$. It routine to check that

$$W(\vec{x}^{(1)}\ \vec{x}^{(2)})(t) = \det\left[\begin{bmatrix}1\\1\end{bmatrix}e^{-t},\ \begin{bmatrix}1\\1\end{bmatrix}te^{-t} + \begin{bmatrix}0\\1/5\end{bmatrix}e^{-t}\right]$$

$$= \det\begin{bmatrix}e^{-t} & te^{-t}\\ e^{-t} & te^{-t} + \frac{1}{5}e^{-t}\end{bmatrix}$$

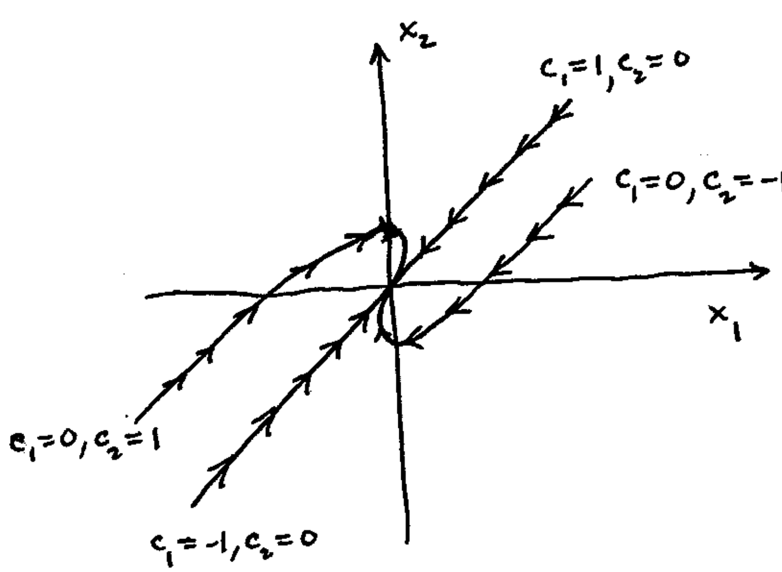
$$= \frac{1}{5}e^{-2t} \neq 0$$

So $\vec{x}^{(1)}(t) = \begin{bmatrix}1\\1\end{bmatrix}e^{-t}$ and $\vec{x}^{(2)}(t) = \begin{bmatrix}1\\1\end{bmatrix}te^{-t} + \begin{bmatrix}0\\1/5\end{bmatrix}e^{-t}$ form a F.S.S. of

$\vec{x}' = \begin{bmatrix}-6 & 5\\-5 & 4\end{bmatrix}\vec{x}$. Hence, the general solution is

$$\vec{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-t} + c_2\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-t} + \begin{bmatrix}0\\1/5\end{bmatrix}e^{-t}\right)$$

where c_1 and c_2 are arbitrary constants. We sketch some typical trajectories in the x_1x_2-plane below.



(c) Note that since $e^{-t} \to 0$ and $te^{-t} \to 0$ as $t \to \infty$, $\vec{x}(t) \to \begin{bmatrix}0\\0\end{bmatrix}$ as $t \to \infty$.

Summary: If λ is a repeated eigenvalue for a 2×2 matrix A and A has only one (linearly independent) eigenvector $\vec{k}$ corresponding to λ, then a fundamental set of solutions for $\vec{x}' = A\vec{x}$ is

$$\vec{x}^{(1)}(t) = \vec{k}e^{\lambda t} \quad \text{and} \quad \vec{x}^{(2)}(t) = \vec{k}te^{\lambda t} + \vec{l}e^{\lambda t}$$

where $(A-\lambda I)\vec{l} = \vec{k}$. (See #17, 18 pp.429f for repeated eigenvalues with multiplicity 3.)

Ex 2 (#6, p.428) Find the general solution of $\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\vec{x}$.

Solution: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and note that $A^T = A$; i.e. A is symmetric.

Therefore A will have 3 linearly independent eigenvectors. Using an HP-49G calculator we find:

	Eigenvalues	Eigenvectors
	$\lambda_1 = 2$	$\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
repeated eigenvalue of multiplicity 2	$\lambda_2 = -1$	$\vec{k}^{(2)} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
	$\lambda_3 = -1$	$\vec{k}^{(3)} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

The general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1\vec{k}^{(1)}e^{\lambda_1 t} + c_2\vec{k}^{(2)}e^{\lambda_2 t} + c_3\vec{k}^{(3)}e^{\lambda_3 t} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-t}.$$