Mathematics 204
Spring 2014
Exam II

Your Printed Name: Solution

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turn off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page and 5 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	**Sum**
Maximum points	20	20	20	20	20	100
Points earned						

1. [20] Find the general solution of the following differential equations:

(a) $y''' - y'' - y' + y = 0$

The characteristic equation: $r^3 - r^2 - r + 1 = 0$

$\Rightarrow (r^3 - r^2) - (r-1) = 0$

$\Rightarrow r^2(r-1) - (r-1) = 0$

$\Rightarrow (r-1)(r^2-1) = 0$

$\Rightarrow (r-1)^2(r+1) = 0$

$\therefore\ r_1 = 1$ repeated ; $r_3 = -1$

General solution: $y = c_1 e^t + c_2 t e^t + c_3 e^{-t}$

#

(b) $t^2 y'' + y = 0$

$r(r-1) + 1 = 0 \Rightarrow r^2 - r + 1 = 0$

$\therefore\ r = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

·General solution:

$$y = c_1 |t|^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln|t|\right) + c_2 |t|^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln|t|\right)$$

#

2. [20] Find the general solution of the differential equation

$$4y'' - 4y' + y = \frac{8e^{t/2}}{t}$$

on the interval $0 < t < \infty$.

First, solve the associated homogeneous equation:

$$4y'' - 4y' + y = 0$$

Characteristic equation: $4r^2 - 4r + 1 = 0$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2} \quad \text{repeated.}$$

$\therefore$ The fundamental set of solution is : $y_1 = e^{t/2}$; $y_2 = te^{t/2}$

Next, we cosider the nonhomogeneous equation and use the variation of parameter method to solve it. Rewrite the equation into the standard form:

$$y'' - y' + \frac{1}{4}y = 2\frac{e^{t/2}}{t} \qquad \therefore \; g = \frac{2e^{t/2}}{t}$$

The wronskian is computed by
$$W = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & (\frac{1}{2}+t)e^{t/2} \end{vmatrix} = \frac{1}{2}e^{t} \neq 0$$

$$u_1 = -\int \frac{y_2\, g}{W}dt = -\int \frac{(te^{t/2})\left(\frac{2e^{t/2}}{t}\right)}{\frac{1}{2}e^t}dt = -\int 4\,dt = -4t + C_1$$

$$u_2 = \int \frac{y_1\, g}{W}dt = \int \frac{(e^{t/2})\left(\frac{2e^{t/2}}{t}\right)}{\frac{1}{2}e^t}dt = \int \frac{4}{t}dt = 4\ln|t| + C_2$$

$$= 4\ln t + C_2 \quad (\because t > 0)$$

General solution of the nonhomogeneous equation is.

$$y = u_1y_1 + u_2y_2 = [-4t + C_1]e^{t/2} + [4\ln t + C_2]te^{t/2}$$

#

3. [20] Use the **method of undetermined coefficients** to find the general solution of the differential equation

$$y'' + 4y' + 13y = 6te^t + 11e^t. \qquad (*)$$

(NOTE: No credit will be awarded for any other method of solution.)

First, solve the associated homogeneous equation:

$$y'' + 4y' + 13y = 0$$

characteristic equation: $r^2 + 4r + 13 = 0$

$$\Rightarrow \quad r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$\Rightarrow$ Complementary solution: $y_c = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$.

The particular solution:

$$Y = (At+B)e^t$$

$$\Rightarrow Y' = Ae^t + (At+B)e^t = [At + (A+B)]e^t$$

$$Y'' = Ae^t + [At + (A+B)]e^t = [At + (2A+B)]e^t$$

Substituting Y, Y' and Y'' into equation $(*)$, we obtain.

$$LHS = [At + (2A+B)]e^t + 4[At + (A+B)]e^t + 13(At+B)e^t$$

$$= 18Ate^t + [6A + 18B]e^t$$

$$= RHS = 6te^t + 11e^t$$

$$\therefore \begin{cases} 18A = 6 & \Rightarrow A = 1/3 \\ 6A + 18B = 11 & \Rightarrow B = \frac{1}{18}(11 - 6A) = \frac{1}{2} \end{cases}$$

4. [20] A mass weighing 8 lb hangs from a vertical spring attached to a rigid support. At its equilibrium position, the mass stretches the spring 4 inches from its natural length. The mass is in a medium with a damping constant of 20 lbs/ft. Suppose the mass is displaced an additional 2 feet in the upward direction and then released. Suppose that the body is acted upon by an external downward force of $0.001\cos(\omega t)$ pounds after it is released. Then

(a) Set up, **BUT DO NOT SOLVE**, an initial value problem that models the motion of the body. (Assume that the acceleration of gravity is 32 ft/s^2.)

Denote $u(t)$ as the displacement of the mass from its equilibrium position.

$$mu'' + \gamma u' + ku = F(t)$$

$$mg = 8\text{ lb} \Rightarrow m = \frac{8\text{ lb}}{32\text{ ft}/s^2} = \frac{1}{4}\,\frac{\text{lb}\cdot s^2}{\text{ft}}$$

$$L = 4\text{ in} = \frac{1}{3}\text{ ft}; \quad mg = kL \Rightarrow k = \frac{mg}{L} = \frac{8}{\frac{1}{3}} = 24\,\frac{\text{lb}}{\text{ft}}$$

$$\gamma = 20\,\frac{\text{lbs}}{\text{ft}}; \qquad F(t) = 0.001\cos(\omega t)$$

Hence, the motion of the mass can be described by

$$\begin{cases} \frac{1}{4}u'' + 20u' + 24u = 0.001\cos(\omega t) \\ u(0) = -2; \quad u'(0) = 0 \end{cases}$$

#

(b) Does there exist a value of the frequency ω which will cause resonance? If yes, find the resonant frequency ω. Otherwise, explain why there is no such frequency.

No ω will cause resonance, because of there is a damping force.

5. [20] Use the definition of the Laplace transform to find the Laplace transform of the function

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le t \le 2. \\ 2, & t > 2. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Laplace transform of $f(t)$ is defined as

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \cancel{\int_0^1 e^{-st} \cdot 0\, dt} + \int_1^2 e^{-st} \cdot 1\, dt + \int_2^\infty e^{-st} \cdot 2\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_1^2 + 2\int_2^\infty e^{-st}\, dt$$

$$= -\frac{1}{s}\left[e^{-2s} - e^{-s}\right] + 2\int_2^\infty e^{-st}\, dt$$

where

$$2\int_2^\infty e^{-st}\, dt = 2\lim_{A\to\infty} \int_2^A e^{-st}\, dt = 2\lim_{A\to\infty}\left[-\frac{1}{s} e^{-st}\Big|_2^A\right]$$

$$= 2\lim_{A\to\infty}\left[-\frac{1}{s}\left(\cancel{e^{-sA}} - e^{-2s}\right)\right]$$

$$= \frac{2}{s} e^{-2s} \qquad (s > 0)$$

Hence, the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f\}(s) = -\frac{1}{s}\left(e^{-2s} - e^{-s}\right) + \frac{2}{s} e^{-2s} = \frac{e^{-2s} + e^{-s}}{s}$$

for $s > 0$

#