Mathematics 204

Fall 2011

Exam I

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 6 pages of problems containing 6 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

	0	1	2	3	4	5	6	Sum
points earned								
maximum points	2	17	16	16	16	17	16	100

1.[17] Find the explicit solution to $(y+t^2y)y' = 2t$ satisfying the initial condition $y(0) = -2$.

$$y(1+t^2)y' = 2t \quad \text{(1st order; variables separable)}$$

$$y\,dy = \frac{2t\,dt}{1+t^2}$$

$$\int y\,dy = \int \frac{2t}{1+t^2}\,dt$$

$$\frac{1}{2}y^2 = \ln(1+t^2) + c_1$$

$$y = \pm\sqrt{c + 2\ln(1+t^2)} \quad (2c_1 = c \text{ is an arbitrary constant})$$

We need to choose the minus sign in order to meet the initial condition $y(0) = -2$:

$$y(t) = -\sqrt{c + 2\ln(1+t^2)}\,.$$

$$\underset{\substack{\uparrow \\ \text{Want}}}{-2} = y(0) = -\sqrt{c + \underbrace{2\ln(1)}_{0}} \quad \text{so we must choose } c = 4.$$

$$\boxed{y(t) = -\sqrt{4 + 2\ln(1+t^2)}}$$

2.[16] Solve the differential equation $\underbrace{t^5 y' + 6t^4 y = e^{-t}}_{\text{1st order; linear equation.}}$ on the interval $t > 0$.

$$y' + \frac{6}{t} y = \frac{e^{-t}}{t^5}$$

Integrating factor: $e^{\int p(t)\,dt} = e^{\int \frac{6}{t}\,dt} = e^{6\ln(t) + \cancel{c}^{\,0}} = e^{\ln(t^6)} = t^6.$

$$t^6\left[y' + \frac{6}{t} y \right] = t^6 \left[\frac{e^{-t}}{t^5} \right]$$

$$\underbrace{t^6 y' + 6t^5 y}_{\text{exact!}} = t e^{-t}$$

$$\frac{d}{dt}\left[t^6 y \right] = t e^{-t}$$

Integrate both sides:

$$t^6 y = \int \frac{d}{dt}\left[t^6 y\right] dt = \int \underbrace{t}_{U} \underbrace{e^{-t} dt}_{dV} = -te^{-t} - \int -e^{-t} dt = -te^{-t} - e^{-t} + c$$

$$\boxed{y(t) = \frac{-(t+1)e^{-t}}{t^6} + \frac{c}{t^6}}$$

where c is an arbitray constant.

3.[16] A tank originally contains 100 gallons of water with 50 pounds of salt dissolved in it. Water containing 2 pounds of salt per gallon is entering the tank at a rate of 4 gallons per minute, and the well-stirred mixture leaves the tank at a rate of 5 gallons per minute. Write, **BUT DO NOT SOLVE,** an initial value problem that models the amount of salt in the tank for times in the interval $0 \le t \le 100$ minutes.

$A(t)$ = amount of salt (in pounds) in the tank at time t (in minutes)

Net rate of change of salt in tank = Rate at which salt enters tank − Rate at which salt leaves tank

$$\frac{dA}{dt} = \left(\frac{4\text{ gal}}{\text{min}}\right)\left(\frac{2\text{ lb.}}{\text{gal}}\right) - \left(\frac{5\text{ gal}}{\text{min}}\right)\left(\frac{A(t)\text{ lbs.}}{V(t)\text{ gal.}}\right)$$

$$\boxed{\begin{array}{l} \dfrac{dA}{dt} = 8 - \dfrac{5A}{100-t} \\[2ex] A(0) = 50 \end{array}}$$

(A in pounds, t in minutes)

t(min)	V(t) gal.
0	100
1	99
2	98
⋮	
t	100 − t

4.[16] Find the general solution of the following differential equations.

(a) $2y''+6y'+5y=0$ $\quad y=e^{rt}$ leads to $2r^2+6r+5=0$, Then the quadratic formula $r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ yields $r=\dfrac{-6\pm\sqrt{36-40}}{4}=\dfrac{-6\pm 2i}{4}=-\dfrac{3}{2}\pm\dfrac{1}{2}i$.

The general solution is $y=e^{\lambda t}\left[c_1\cos(\mu t)+c_2\sin(\mu t)\right]$ where $\lambda=-\frac{3}{2}$ and $\mu=\frac{1}{2}$.

$$\therefore \boxed{y(t)=e^{-\frac{3}{2}t}\left[c_1\cos\left(\tfrac{t}{2}\right)+c_2\sin\left(\tfrac{t}{2}\right)\right]}$$

where c_1, c_2 are arbitrary constants.

(b) $4y''-20y'+25y=0$ $\quad y=e^{rt}$ leads to $4r^2-20r+25=0$ or $(2r-5)^2=0$ so $r=\frac{5}{2}$ with multiplicity two. The general solution is

$$\boxed{y(t)=c_1e^{\frac{5}{2}t}+c_2te^{\frac{5}{2}t}}$$

where c_1, c_2 are arbitrary constants.

5.[17] Find the general solution of the differential equation $y'' - y' - 2y = 8e^{3t}$. $(*)$

$y = e^{rt}$ in $y'' - y' - 2y = 0$ leads to $r^2 - r - 2 = 0$ or $(r-2)(r+1) = 0$ so $r = 2$ or $r = -1$. Therefore $y_c(t) = c_1 e^{2t} + c_2 e^{-t}$ is the complementary solution of $(*)$; i.e. the general solution of the associated homogeneous equation $y'' - y' - 2y = 0$.

The method of undetermined coefficients suggests a trial particular solution of $(*)$ of the form $y_p(t) = Ae^{3t}$ where A is a constant to be determined so y_p solves $(*)$.

Then $y_p' = 3Ae^{3t}$ and $y_p'' = 9Ae^{3t}$ so $y_p'' - y_p' - 2y_p = 8e^{3t}$ is equivalent to

$9A\cancel{e^{3t}} - 3A\cancel{e^{3t}} - 2A\cancel{e^{3t}} = 8\cancel{e^{3t}}$ and hence $4A = 8$ so $A = 2$. Thus

$y_p(t) = 2e^{3t}$ is a particular solution of $(*)$. The general solution of $(*)$ is

$y = y_c + y_p$ or $\boxed{y(t) = c_1 e^{2t} + c_2 e^{-t} + 2e^{3t}}$ where c_1, c_2 are arbitrary constants.

6.[16] Given that $y_1(t) = t^2$ is a solution of the differential equation $t^2 y'' - 3ty' + 4y = 0$, use reduction of order to find a second linearly independent solution on the interval $t > 0$.

Assume $y_2(t) = u(t)y_1(t) = u(t)t^2$ is a second solution of the DE where $u = u(t)$ is a function to be determined so that

$$(*) \qquad t^2 y_2''(t) - 3t y_2'(t) + 4y_2(t) = 0 \qquad \text{for all } t > 0.$$

Note that $y_2' = t^2u' + 2tu$ and $y_2'' = t^2u'' + 2tu' + 2tu' + 2u = t^2u'' + 4tu' + 2u$.
Substituting these expressions for y_2 and its derivatives in $(*)$ yields

$$t^2\left(t^2u'' + 4tu' + 2u\right) - 3t\left(t^2u' + 2tu\right) + 4ut^2 = 0.$$

Simplifying, we have

$$t^4u'' + (4t^3 - 3t^3)u' + (2t^2 - 6t^2 + 4t^2)u = 0$$

$$t^4u'' + t^3u' = 0.$$

Let $u' = v$. Then $u'' = v'$ so the above equation is equivalent to

$$tv' + v = 0.$$

The left member is exact (so we don't need an integrating factor for this 1st order linear DE):

$$\frac{d}{dt}\left[tv\right] = 0.$$

Integrating both sides yields

$$tv = c_1.$$

Hence

$$u' = v = \frac{c_1}{t}$$

so integrating again yields $u = c_1 \ln(t) + c_2$. Consequently $y_2(t) = u(t) \cdot t^2 = (c_1 \ln(t) + c_2)t^2 = c_1 t^2 \ln(t) + c_2 t^2$. Take $c_1 = 1$ and $c_2 = 0$ to get a solution that is linearly independent from y_1, $\boxed{y_2(t) = t^2 \ln(t)}$, on $t > 0$.

Check: $W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^2\ln(t) \\ 2t & t + 2t\ln(t) \end{vmatrix} = t^3 \neq 0$ on $t > 0$.

2011 Fall Semester, Math 204 Hour Exam I, Master List

100	卌 卌 IIII	59	I	19	
99	卌 卌 卌 卌	58	卌 I	18	
98	卌 卌 IIII	57	III	17	
97	卌 卌 卌 卌	56	卌	16	
96	卌 卌 卌 卌 卌	55	II	15	
95	卌 卌 卌 卌 III	54	卌	14	
94	卌 卌 卌 II	53	III	13	
93	卌 卌 II	52	I	12	
92	卌 卌 III	51	I	11	
91	卌 卌 卌	50	III	10	
90	卌 IIII	49	IIII	9	
89	卌 II	48	II	8	
88	卌 卌 I	47	II	7	
87	卌 IIII	46	IIII	6	
86	卌 卌 卌	45	I	5	
85	卌 卌	44	II	4	
84	卌 卌 卌 II	43	I	3	
83	卌 卌 III	42	I	2	
82	卌 IIII	41		1	
81	卌 IIII	40		0	
80	卌 IIII	39			
79	卌 IIII	38			
78	IIII	37	I		
77	III	36	I		
76	卌 I	35	II		
75	卌 IIII	34	III		
74	卌 I	33			
73	卌	32	I		
72	IIII	31			
71	卌 II	30	I		
70	卌 II	29			
69	卌 I	28	I		
68	卌 III	27	I		
67		26	I		
66	卌	25			
65	III	24	I		
64	II	23	I		
63	III	22	I		
62	卌	21			
61	II	20			
60	卌				

182 A's

109 B's

60 C's

39 D's

62 F's

Number taking exam: 452
Median: 85
Mean: 80.51
Standard Deviation: 17.60

Number receiving A's: 182 — 40.3%
Number receiving B's: 109 — 24.1
Number receiving C's: 60 — 13.3
Number receiving D's: 39 — 8.6
Number receiving F's: 62 — 13.7