Mathematics 204

Fall 2012

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**

2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.

3. You are **not allowed to use a calculator** on this exam.

4. Exam II consists of this cover page, 6 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.

5. Once the exam begins, you will have 60 minutes to complete your solutions.

6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.

7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.

8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

problem	1	2	3	4	5	6	Sum
points earned							
maximum points	17	17	17	17	16	16	100

1.[17] Find the general solution of each differential equation.

(a) $y^{(4)} + 4y'' + 4y = 0$ $\quad y = e^{rt}$ leads to $r^4 + 4r^2 + 4 = 0$ or $(r^2+2)^2 = 0$ with roots $r = \sqrt{2}i$ (multiplicity 2) and $r = -\sqrt{2}i$ (multiplicity 2). Therefore

$$\boxed{y = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + c_3 t\cos(\sqrt{2}t) + c_4 t\sin(\sqrt{2}t)}$$

is the general solution. Here $c_1, c_2, c_3,$ and c_4 are arbitrary constants.

(b) $t^2 y'' - 6y = 0$ $\quad$ (Euler equation) $y = t^m$ leads to $m(m-1) - 6 = 0$ or $m^2 - m - 6 = 0$ or $(m-3)(m+2) = 0$ with roots $m = -2$ and $m = 3$. Therefore

$$\boxed{y = c_1 t^{-2} + c_2 t^3}$$

is the general solution on any interval that does not contain $t = 0$. Here c_1 and c_2 are arbitrary constants.

2.[17] Find the general solution of $y'' + 6y' + 9y = \dfrac{e^{-3t}}{t^3}$.

$y = e^{rt}$ in $y'' + 6y' + 9y = 0$ leads to $r^2 + 6r + 9 = 0$ or $(r+3)^2 = 0$ with roots $r = -3$ (multiplicity 2). Thus $y_c(t) = c_1 e^{-3t} + c_2 t e^{-3t}$. Note that

$$W(e^{-3t}, te^{-3t}) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{vmatrix} = e^{-6t}(1-3t) + 3te^{-6t} = e^{-6t} \neq 0 \text{ so}$$

$y_1(t) = e^{-3t}$, $y_2(t) = te^{-3t}$ form a fundamental set of solutions to $y'' + 6y' + 9y = 0$.

We use variation of parameters to write a particular solution of the nonhomogeneous DE:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = e^{-3t}u_1(t) + te^{-3t}u_2(t)$$

where

$$u_1(t) = \int \frac{-y_2 g}{W}\,dt = \int \frac{-te^{-3t}\cdot e^{-3t}t^{-3}}{e^{-6t}}\,dt = \int -t^{-2}\,dt = t^{-1} + \cancel{c}^{\,0}$$

$$u_2(t) = \int \frac{y_1 g}{W}\,dt = \int \frac{e^{-3t}\cdot e^{-3t}t^{-3}}{e^{-6t}}\,dt = \int t^{-3}\,dt = \frac{t^{-2}}{-2} + \cancel{c}^{\,0}.$$

Hence

$$y_p(t) = t^{-1}e^{-3t} - \tfrac{1}{2}t^{-2}\cdot te^{-3t} = \tfrac{1}{2}t^{-1}e^{-3t}.$$

The general solution is

$$y = y_c + y_p$$

$$\boxed{y(t) = c_1 e^{-3t} + c_2 te^{-3t} + \tfrac{1}{2}t^{-1}e^{-3t}}.$$

3.[17] Solve the initial value problem $y''' - 2y'' - y' + 2y = 2t - 1$, $y(0) = 0$, $y'(0) = -2$, $y''(0) = -3$.

$y = e^{rt}$ in $y''' - 2y'' - y' + 2y = 0$ leads to $r^3 - 2r^2 - r + 2 = 0$. By inspection, $r = 1$ is a root.

$$
\begin{array}{r|l}
 & r^2 - r - 2 \\
\hline
r - 1 & r^3 - 2r^2 - r + 2 \\
 & -(r^3 - r^2) \\
 & \quad -r^2 - r \\
 & \quad -(-r^2 + r) \\
 & \qquad -2r + 2 \\
 & \qquad -(-2r + 2) \\
 & \qquad\quad 0 = R
\end{array}
$$

Therefore, the characteristic equation factors as $(r-1)(r^2 - r - 2) = 0$ or $(r-1)(r-2)(r+1) = 0$ with roots $r = 1$, $r = -1$, and $r = 2$. Hence $y_c(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}$. We use the method of undetermined coefficients to find a particular solution. Since $g(t) = 2t - 1$ is a polynomial of degree 1 and $\alpha = 0$ is not a root of the characteristic equation, a trial form for the particular solution is $y_p = At + B$ where A and B are constants to be determined so y_p solves the nonhomogeneous DE: $y_p''' - 2y_p'' - y_p' + 2y_p = 2t - 1$ or equivalently $0 - 2\cdot 0 - A + 2(At + B) = 2t - 1$ so $2A = 2$ and $2B - A = -1$. That is, $A = 1$ and $B = 0$. Thus $y_p = t$ and $y = y_c + y_p = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + t$. To apply the initial conditions, we need $y' = c_1 e^{t} - c_2 e^{-t} + 2c_3 e^{2t} + 1$ and $y'' = c_1 e^{t} + c_2 e^{-t} + 4c_3 e^{2t}$

Therefore

$$
\begin{cases}
0 = y(0) = c_1 + c_2 + c_3 \\
-2 = y'(0) = c_1 - c_2 + 2c_3 + 1 \\
-3 = y''(0) = c_1 + c_2 + 4c_3
\end{cases}
\Leftrightarrow
\begin{cases}
0 = c_1 + c_2 + c_3 \\
-3 = c_1 - c_2 + 2c_3 \\
-3 = c_1 + c_2 + 4c_3 .
\end{cases}
$$

Adding the first equation to the second and adding -1 times the first equation to the third yields

$$
\begin{cases}
0 = c_1 + c_2 + c_3 \\
-3 = 2c_1 \quad + 3c_3 \\
-3 = \qquad\quad + 3c_3
\end{cases}
\Leftrightarrow
\begin{cases}
c_2 = 1 \\
c_1 = 0 \\
c_3 = -1 .
\end{cases}
$$

Therefore

$$\boxed{y(t) = e^{-t} - e^{2t} + t}$$

solves the IVP.

Alternate solution of #3 via Laplace transforms:

$$\mathcal{L}\{y''' - 2y'' - y' + 2y\}(s) = \mathcal{L}\{2t-1\}(s)$$

$$s^3\mathcal{L}\{y\}(s) - s^2\overset{0}{\cancel{y(0)}} - s\overset{-2}{\cancel{y'(0)}} - \overset{-3}{\cancel{y''(0)}} - 2\left(s^2\mathcal{L}\{y\}(s) - s\overset{0}{\cancel{y(0)}} - \overset{-2}{\cancel{y'(0)}}\right) - \left(s\mathcal{L}\{y\}(s) - \overset{0}{\cancel{y(0)}}\right) + 2\mathcal{L}\{y\}(s) = \frac{2}{s^2} - \frac{1}{s}$$

$$(s^3 - 2s^2 - s + 2)\mathcal{L}\{y\}(s) = -2s - 3 + 4 + \frac{2}{s^2} - \frac{1}{s}$$

$$\mathcal{L}\{y\}(s) = \frac{(-2s+1)s^2 + 2 - s}{(s^3 - 2s^2 - s + 2)s^2} = \frac{-2s^3 + s^2 - s + 2}{(s-1)(s+1)(s-2)s^2} = \frac{\cancel{(s-1)}(-2s^2 - s - 2)}{s^2\cancel{(s-1)}(s+1)(s-2)}.$$

$$\frac{-2s^2 - s - 2}{s^2(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-2}$$

$$\Rightarrow \quad -2s^2 - s - 2 = As(s+1)(s-2) + B(s+1)(s-2) + Cs^2(s-2) + Ds^2(s+1)$$

To find B, set $s=0$: $\quad -2 = B(1)(-2) \Rightarrow B = 1$

To find C, set $s=-1$: $\quad -3 = C(1)(-3) \Rightarrow C = 1$

To find D, set $s=2$: $\quad -12 = D(4)(3) \Rightarrow D = -1$

To find A, set $s=1$: $\quad -5 = A(-2) + B(-2) + C(-1) + D(2) \Rightarrow -5 = -2A - 2 - 1 - 2$

$\Rightarrow A = 0$

Therefore $\mathcal{L}\{y\}(s) = \dfrac{1}{s^2} + \dfrac{1}{s+1} - \dfrac{1}{s-2}$

so $y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2} + \dfrac{1}{s+1} - \dfrac{1}{s-2}\right\}$

$$\boxed{y(t) = t + e^{-t} - e^{2t}}.$$

4.[17] [In the following problem, assume that the acceleration of gravity is 9.8 meters per second per second.] A 5 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 20 centimeters beyond its natural length. The body is acted upon by a downward external force of $10\sin(t/2)$ newtons and there is no damping.

14 (a) If the body is set in motion from a position 10 centimeters below its equilibrium position with an upward initial velocity of 30 centimeters per second, set up, **BUT DO NOT SOLVE**, an initial value problem that describes the motion of the body.

Let $y(t)$ denote the vertical displacement of the body from its static equilibrium position at time t (with y in meters and t in seconds). Then $my'' + \gamma y' + ky = g(t)$ where $m = 5$ kg, $\gamma = 0$, $g(t) = 10\sin(t/2)$, and $mg = ks_0$ so $k = \frac{mg}{s_0} = \frac{5(9.8)}{0.2} = 245$.

Therefore

$$\boxed{5y'' + 245y = 10\sin(t/2), \quad y(0) = 0.1, \quad y'(0) = -0.3,}$$

is an IVP that models the body's motion.

(b) If the given downward external force is replaced by $4\cos(\omega t)$ newtons, find the value of the frequency ω which will cause resonance or explain why there is no such frequency.

$$5y'' + 245y = 4\cos(\omega t).$$

Since $\gamma = 0$, resonance is possible. Then $y = e^{rt}$ in $5y'' + 245y = 0$ leads to $5r^2 + 245 = 0$ or $r^2 = -49$ so $r = \pm 7i$. The freely oscillating system has solution $y(t) = c_1\cos(7t) + c_2\sin(7t)$ so $\omega_0 = 7$ is the natural frequency of the system. The condition for resonance is natural frequency = driver frequency; i.e. $\boxed{7 = \omega}$.

5.[16] Use the definition of the Laplace transform,

$$\mathcal{L}\{f\}(s)=\int_0^\infty f(t)e^{-st}\,dt$$

for those values of s for which the improper integral converges, to find the Laplace transform of the function $f(t)=te^{at}$ where a is a real constant. For which values of s is the Laplace transform of f defined?

$$\mathcal{L}\{f\}(s) = \lim_{M\to\infty}\int_0^M te^{at}\cdot e^{-st}\,dt = \lim_{M\to\infty}\int_0^M \underbrace{t}_{u}\,\underbrace{e^{t(a-s)}\,dt}_{dv} \quad \left(\text{Integrate by parts.}\right)$$

$$= \lim_{M\to\infty}\left\{ \left.\frac{te^{t(a-s)}}{a-s}\right|_{t=0}^{M} - \int_0^M \frac{e^{t(a-s)}}{a-s}\,dt \right\}$$

$$= \lim_{M\to\infty}\left\{ \frac{M}{(a-s)e^{M(s-a)}} - \frac{1}{(a-s)^2}\left[e^{M(a-s)} - 1\right]\right\}.$$

We need $s-a>0$ in order for the limit above to exist. If $s>a$ then

$$\lim_{M\to\infty}\frac{M}{e^{M(s-a)}} \overset{\text{L'Hospital}}{=} \lim_{M\to\infty}\frac{1}{(s-a)e^{M(s-a)}} = 0$$

and

$$\lim_{M\to\infty} e^{M(a-s)} = \lim_{M\to\infty}\frac{1}{e^{M(s-a)}} = 0.$$

Therefore $\boxed{\mathcal{L}\{f\}(s) = \frac{1}{(s-a)^2}}$ if $\boxed{s>a.}$

6.[16] Find the inverse Laplace transform of $F(s)=\frac{s^2+s+2}{s^3+s}$.

The partial fraction decomposition of $F(s)$ is $\frac{s^2+s+2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$.

Hence $s^2+s+2 = A(s^2+1)+(Bs+C)s$.

To find A, set $s=0$: $2 = A$.

To find B and C, set $s=i$: $-1+i+2 = A(0)+(Bi+C)i$

$$1+i = -B+Ci$$

Therefore $1=-B$ and $1=C$ and hence

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{-s+1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right\}$$

so

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = 2-\cos(t)+\sin(t)}.$$

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f\}(s) = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$

2012 Fall Semester, Math 204 Hour Exam II, Master List

100	卌 |||		59	|||		19	
99	卌			58	卌						18	
98	卌 卌		57	||		17	
97	卌 ||		56	|||		16	
96	卌			86 As ✓	55						102 Fs ✓	15		
95	卌			54	卌			14		
94	卌 ||		53	|||		13	
93	卌						52	卌			12	
92	卌			51							11	
91	卌 卌		50	卌			10	
90	卌 卌		49	||||		9	
89	卌					48	卌			8	
88	卌 卌 ||		47	|||		7	
87	卌 |||		46	|||		6	
86	卌		45					5		
85	卌 卌 ||	85 Bs ✓	44	||||		4	
84	卌			43			3	
83	卌			42						2	
82	卌 卌		41	卌		1	
81	卌 |||		40			0	
80	卌 卌		39	||			
79	卌			38					
78							37				
77	卌				36					
76	卌 卌		35	卌			
75	卌 ||||	69 Cs ✓	34				
74	卌 卌		33					
73	卌				32					
72	卌 ||		31	||			
71	卌		30					
70							29					
69					28					
68					27							
67	卌				26					
66	卌			25				
65	卌			24				
64	|||	40 Ds ✓	23				
63	卌		22				
62							21						
61						20					
60									

Number taking exam: 382 ✓
Median: 76
Mean: 72.5
Standard Deviation: 19.8

Number receiving A's:	86	22.5%
Number receiving B's:	85	22.3
Number receiving C's:	69	18.1
Number receiving D's:	40	10.5
Number receiving F's:	102	26.7

2012 Fall Semester, Math 204 Hour Exam II

Instructor Dr. Grow, Section M

100			59			19
99	I		58	II		18
98	II		57			17
97	III		56			16
96	I		55	I		15
95		12 As	54			14
94			53			13
93	II		52			12
92	II		51	I	8 Fs	11
91	I		50			10
90			49			9
89			48	I		8
88			47			7
87			46			6
86	I		45	I		5
85		5 Bs	44			4
84	I		43			3
83			42	I		2
82	I		41			1
81	II		40			0
80			39			
79	I		38			
78	I		37			
77	II		36			
76			35			
75		7 Cs	34			
74	I		33			
73			32			
72	I		31			
71			30			
70	I		29	I		
69			28			
68			27			
67	I		26			
66	I		25			
65	I	6 Ds	24			
64			23			
63			22			
62	I		21			
61	I		20			
60	I					

Number taking exam: 38

Median: 77.5

Mean: 75.0

Standard Deviation: 18.5

Number receiving A's: 12 — 31.6%

Number receiving B's: 5 — 13.2

Number receiving C's: 7 — 18.4

Number receiving D's: 6 — 15.8

Number receiving F's: 8 — 21.1